THE HISTORICAL EMERGENCE AND THEORETICAL FOUNDATIONS OF QUADRATIC STOCHASTIC OPERATORS.

Yashiyeva Feruza Yusuf qizi

PhD of Bukhara State University,

Teacher of the Department of Mathematics and Informatics

http://orcid.org/0000-0001-8740-6617

yashiyevaferuza@gmail.com

Abstract: Quadratic Stochastic Operators (QSOs) have played a pivotal role in the development of nonlinear dynamical systems theory, particularly within the context of population genetics and evolutionary processes. Originating from the early 20th century works on heredity and gene frequency models, QSOs generalize classical linear stochastic operators by incorporating quadratic terms that reflect pairwise interactions in populations. This paper reviews the historical emergence of QSOs, tracing their evolution from foundational biological problems to sophisticated mathematical frameworks.

Keywords: Quadratic Stochastic Operators (QSO), Nonlinear operators, Population genetics, Dynamical systems, Evolutionary models, Markov processes, Fixed points, Stability analysis, Ergodic theory, Measure-preserving transformations

1. Introduction. Let us begin by defining a dynamical system. A dynamical system is a mathematical model that describes the evolution of a real-world system—be it physical, biological, economic, or otherwise—where the system's state at any moment is determined by its initial condition.The governing laws of a dynamical system's evolution can take various forms. These include linear or nonlinear differential equations, discrete mappings, graph theory, Markov chain theory, nonequilibrium thermodynamics, chaos theory, synergetics, and others.

It is important to clarify the following concepts in this context:

- **Nonequilibrium thermodynamics** focuses on systems that are away from thermal equilibrium, studying irreversible processes that cannot be described by classical thermodynamics alone.
- **Chaos theory** deals with the behavior of certain nonlinear dynamical systems, particularly those that exhibit extreme sensitivity to initial conditions, leading to what is known as chaotic behavior.
- **Synergetics**, derived from the Greek word meaning "cooperation" or "working together," is an interdisciplinary field that investigates how complex structures and patterns emerge in open systems that are far from equilibrium, often through self-organization and interaction between subsystems.

Depending on the nature of the process being studied, dynamical systems are generally classified into discrete-time and continuous-time systems. In discrete-time systems, traditionally referred to as cascades, the system's behavior—or equivalently, its trajectory in the phase space—is represented by a sequence of states indexed by discrete time steps.

2. Main Part. In contrast, flows refer to continuous-time dynamical systems, where the system's state is defined at every instant of time. Cascades and flows are central objects of study in both symbolic and topological dynamics.A continuous-time dynamical system is often described by an autonomous system of differential equations defined on a certain domain, satisfying the conditions of existence and uniqueness theorems. The equilibrium points of such a system correspond to the singular points of the differential equations, while closed phase curves represent their periodic solutions.

One of the primary objectives of dynamical systems theory is to analyze the curves defined by these differential equations. This includes partitioning the phase space into trajectories and examining their qualitative behavior—such as identifying and characterizing equilibrium states, as well as determining attracting and repelling sets (also known as attractors and repellers).Modern dynamical systems theory finds extensive applications across diverse fields of mathematics and serves as a unifying framework for several branches, including **topology**, **algebra**, **algebraic geometry**, **measure theory**, and the **theory of differential forms**.As previously mentioned, one of the main goals of dynamical system analysis is to understand the **evolution of the system's state** over time. The system's behavior is usually governed by certain deterministic or probabilistic laws. In the context of **mathematical genetics**, such laws are often modeled using **quadratic stochastic operators (QSOs)**.

The term **stochastic** (derived from the ancient Greek word *stokhos*, meaning "aim" or "guess") refers to randomness. A stochastic process is a non-deterministic process in which the future state of the system is influenced by both predictable rules and random variables.The use of the term **stochasticity** in mathematics is closely related to the works of Vladislav Bortkevich, a Russian economist and statistician, who explored hypotheses involving probabilistic approaches.The recurrence of nonlinear changes in various fields has led to the necessity of studying their **ergodic** and **asymptotic** properties. For example, problems in physics involving interactions of multiplying and diffusing particles; biological issues concerning the dynamics of closed genetic populations; and economic problems related to stability in models of collective behavior, among others.In ergodic theory, the repetition of nonlinear transformations, coupled with probabilistic-statistical concepts, leads to the development of ideas related to **invariant measures** and **dynamical systems**.

Specifically, in biology, the mathematical modeling of population evolution is expressed through **quadratic stochastic operators** (QSOs). Within this framework, the problem of finding a stable distribution of various types of individuals in a closed biological system during the evolutionary process corresponds to studying the asymptotic behavior of quadratic stochastic operators.The theory of quadratic stochastic operators attracts considerable mathematical interest due to the abundance of both classical and nonstandard problems, many of which remain unsolved.The investigation of the trajectories—i.e., the sequences of iterations—of quadratic stochastic operators was first addressed in the works of **S. Ulam** and his collaborators. Additionally, numerous numerical analyses of trajectories for various types of quadratic stochastic operators defined on the two-dimensional simplex S^2have been conducted using computer simulations.Subsequent research by S. Ulam and colleagues focused on estimating **Lipschitz constants**, which are essential for the study of quadratic stochastic operators defined on the (n−1)-dimensional simplex S^{n-1}.

3.Discussion part. The Dynamics and Fixed Points of Quadratic Stochastic Operators. $x(t), y(t)$ an autonomous system of differential equations for functions is defined as the following system of differential equations:

$$\frac{dx}{dt} = P(x,y), \frac{dy}{dt} = Q(x,y) \qquad (1)$$

In this case, the right-hand side is independent of the variable t. $x = f(t), y = g(t)$ this is solutions of (1) equals.(1) The phase trajectory of a system is defined as a curve in the $\mathbb{R}^2_{x,y}$ plane given parametrically by $x = f(t), y = g(t)$. The direction of motion along this trajectory is assumed to increase in the same sense as the clockwise rotation of a clock hand.The phase portrait of the system refers to the graphical representation of the phase curves. An equilibrium point, or a stationary point of the autonomous system described by differential equations (1), is a solution of the form $x = x_0, y = y_0$, where the system remains at rest. Recall that an equilibrium trajectory corresponds to a point and $P(x_0, y_0) = Q(x_0, y_0) = 0$.

Simply put, if P and Q are linear functions, then $P(x,y) = ax + by$, $Q(x,y) = cx + dy$, in this a, b, c, d – Constants included, the system can be expressed in the following form.

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy \qquad (2)$$

(2) Using the given coefficients in the system, we construct the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Finding the equilibrium point

The procedure for determining the equilibrium point is as follows:

1. First, we find the roots $\lambda_{1,2}$ of the characteristic equation below.

$$\det(A - \lambda E) = 0$$

(2) The solution of the system takes the following form.

$$\binom{x}{y} = C_1\vec{h}_1 e^{\lambda_1 t} + C_2\vec{h}_2 e^{\lambda_2 t},$$

in this, $\vec{h}_1$ va $\vec{h}_2 - \lambda_1, \lambda_2$ Characteristic vectors that correspond to specific eigenvalues.$\vec{h}_1$and $\vec{h}_2$ The coordinates of points with respect to the basis formed by eigenvectors. We denote it as $\zeta_1 = C_1 e^{\lambda_1 t}, \quad \zeta_2 = C_2 e^{\lambda_2 t}$

2. If the roots are real and distinc $(\lambda_1 \neq \lambda_2)$ and have the same sign $(\lambda_1 \lambda_2 > 0)$ then the equilibrium state is called a node. The node is said to be stable if $\lambda_1 > 0, \lambda_2 < 0$ then it is called unstable if $\lambda_1 > 0, \lambda_2 > 0$ if this is the case. For clarity, we note that $0 < |\lambda_1| < |\lambda_2|$. $\lambda_2 < \lambda_1 < 0$ in this case, both coordinates tend to zero as t increases, but the second coordinate approaches zero more rapidly. That is, all points of the integral trajectory $C_1 = 0$ In this case, both coordinates tend to zero as ttt increases, but the second coordinate approaches zero more quickly. That is, all points on the integral trajectory—except for the one corresponding to the unique solution—approach the origin. The entire trajectory has a common tangent in such a way that $\vec{h}_1$ which is parallel to the vector corresponding to the eigenvalue

λ1\lambda_1λ1 with the smaller absolute value. Moreover, (2) The system of equations has phase trajectories in the form of rays whose points approach the origin (equilibrium state). These rays correspond to the eigenvectors of the matrix A $\vec{h}_1$ and $\vec{h}_2$ are aligned parallel to the eigenvectors. In $t \to -\infty\ to\ \vec{h}_2$ Coordinates parallel to the vector move away from the origin and approach straight lines. In the case of a stable node, motion along the phase trajectory occurs toward the equilibrium state.

$\lambda_2 > \lambda_1 > 0$ In this case, the nature of the trajectory's positioning is completely preserved. ($t \to \infty$ by analogy, ttt is directly $-\infty$ one can reason in a similar way). In the case of an unstable node, motion along the phase trajectory occurs away from the equilibrium state.

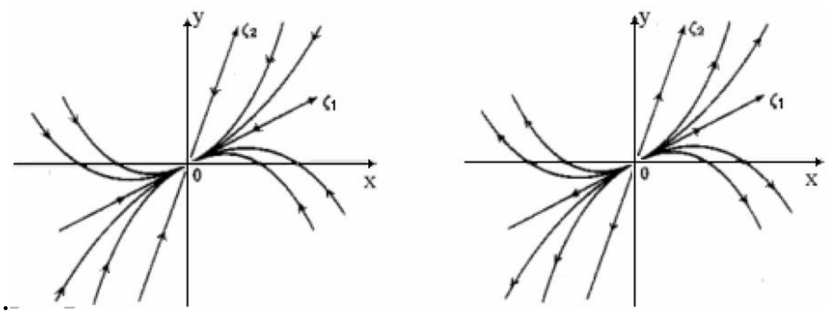

Fig.1

(2) Spatial portrait diagram of the system $a)\ \lambda_2 < \lambda_1 < 0$ va $b)\ 0 < \lambda_1 < \lambda_2$ In cases like this, as illustrated in Figure 1. Here and in the following sections $\zeta_1$and ζ_2 – A These are the coordinates of points in the basis formed by the eigenvectors of the matrix. If the eigenvalues have different signs, ($\lambda_1\lambda_2 < 0$), An object is said to be in a state of equilibrium $\lambda_1 < 0 < \lambda_2$. $C_2 = 0$ As time t increases, the points on the trajectory approach the origin coordinates. $\vec{h}_1$ The rays parallel to the x-axis approach the origin along the coordinate lines. In $C_1 = 0$ starting from the origin of the coordinates $\vec{h}_2$ They move away from it along rays parallel to . These trajectories, called separatrices $t \to -\infty$ and $t \to +\infty$ to suitable for $C_1 \neq 0$ and $C_2 \neq 0$ It corresponds to the asymptotic trajectories aligned with ga. Movement along the

trajectory $C_1 \neq 0$ and in $C_2 \neq 0$ $0\zeta_1$ and $0\zeta_2$ It corresponds to motion along the asymptotes. The phase portrait diagram of system (2) in this case is shown in Figure 2.

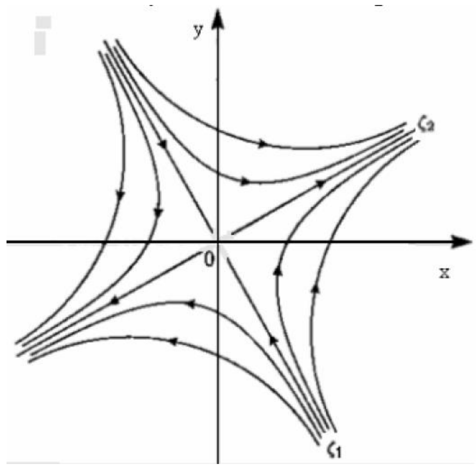

Fig.2.

Note: In the case of a focus, it is not required to find the eigenvectors of matrix A. Apart from the equilibrium states discussed above, several other emergent states that we do not consider can be distinguished.

In conclusion, Quadratic stochastic operators (QSOs) have a rich historical background rooted in the study of nonlinear dynamical systems, particularly in population genetics and evolutionary biology. These operators provide a powerful framework for modeling complex interactions in systems where future states depend quadratically on current distributions. The theoretical foundations of QSOs incorporate ideas from ergodic theory, measure-preserving transformations, and nonlinear analysis, enabling rigorous study of fixed points, stability, and long-term behavior of the system. Understanding the emergence and mathematical properties of QSOs not only advances pure mathematical research but also offers significant insights into real-world phenomena such as gene frequency evolution and species dynamics. Future work continues to explore the diverse applications and generalizations of QSOs in both theoretical and applied contexts.

REFERENCES

[1] Roziqov U.A., Jamilov U.U. Volterrovskiy KCO Dvupoloy populyasii // Ukraiskiy matematicheskiy jurnal, 63:17 (2011), s.985-998.

[2] Rasulov X.R.,Yashiyeva F.Y. O nekotorix vol'terrovskix kvadratichnix stoxasticheskix operatorax dvupoloy populyatsii s neprerivnim vremenem // Nauka, texnika I obrazovaniye, 72:2-2 (2021) s.23-26.

[3] Rasulov X.R.,Yashiyeva F.Y.Ikki jinsli populyatsiyaning dinamikasi haqida // *Scientific progress*, 2:1 (2021), p.665-672.

[4] Rasulov X.R.,Yashiyeva F.Y. Ob odnom kvadratichno stoxasticheskom operatore s neprerivnim vremenem // *«The XXI Century Skills for Professional Activity» International Scientific-Practical Conference*, Tashkent, mart 2021 y., p.145-146.

[5] Rasulov X.R.,Rashidov A.Sh. Organizatsiya prakticheskogo zanyatiya na osnove innovatsionnix texnologiy na urokax matematiki // Nauka, texnika I obrazovaniye, 72:8 (2020) s.29-32.

[6] Rasulov T.H., Rasulov X.R.,O'zgarishi chegaralangan funksiyalar bo'limini o'qitishga doir metodik tavsiyalar // Scientific progress, 2:1 (2021), p.559-567.

[7] Rasulov X.R.,Rashidov A.Sh. O sushestvovanii obobshennogo resheniya krayevoy zadachi dlya nelineynogo uravneniya smeshannogo tipa // Vestnik nauki I obrazovaniya, 97:19-1 (2020), s. 6-9.

[8] Rasulov X.R.,I.dr. O razrazreshimosti zadachi Koshi dlya virojdayushegosya kvazilineynogo uravneniya giperbolicheskogo tipa // Ucheniye XXI veka, mejdunarodniye nauchniy jurnal, 53:6-1 (2019), s.16-18 .

[9] Rasulov X.R., Ob odnoy krayevoy zadache dlya uravneniya giperbolicheskogo tipa // "Komleks analiz, Matematicheskaya Fizika I nelineyniy uraveiya" Mejdunarodnaya nauchnaya konferensiya Sbornik tezisov Bashkortostan RF(oz.Banoy, 18-20 marta 2019 g.), s.65-66.

[10] Rasulov X.R.,Raupova M.X. Rol' matemtiki v biologicheskix naukax // Problemi pedagogiki, № 53:2 (2021), s. 7-10.

[11] Rasulov X.R.,Raupova M.X. Matematicheskiye modeli I zakoni v biologii // *Scientific progress*, 2:2 (2021), p.870-879.

[12] Lakaev S.N., Rasulov T.Kh. (2003). A Model in the Theory of Perturbations of the Essential Spectrum of Multiparticle Operators. *Math. Notes.* 73:4, pp. 521-528.