FIZIK JARAYONLARNI MODELLASHTIRISHDA DIFFERENSIAL TENGLAMALARNING SONLI YECHIMI VA ALGORITMLARI

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Annatatsiya: Fizik jarayonlarni modellashtirishda differensial tenglamalarning sonli yechimi va algoritmlari jarayonning murakkabligini hisobga olgan holda analitik echimni topish qiyin bo'lgan hollarda ishlatiladi. Bu jarayonni diskret nuqtalarga bo'lish orqali amalga oshiriladi, shundan so'ng ushbu nuqtalardagi qiymatlarni hisoblash uchun turli xil algoritmlar, masalan,Neyman usuli,Eyler usuli va Runge-kutta usuli qo'llaniladi.Algoritmlarni tanlash hisoblash quvvati,aniqlik talablariga bog'liq bo'ladi.

Kalit so‘zlar: Fizik jarayonlarni modellashtirish, differensial tenglamalar, Neyman usuli,Eyler usuli va Runge-kutta usuli.

Eyler usuli: Eng sodda va eng tezkor usullardan biri bo'lib, lekin aniqligi past bo'lishi mumkin. U bir nuqtadagi qiymatni keyingi nuqtadagi qiymatni hisoblash uchun ishlatadi.

Neyman usuli: To'lqin jarayonlari va boshqa chekli differensial tenglamalar uchun ishlatiladi.

Runge-kutta usuli: Ko'proq aniqlikni ta'minlaydi, ammo hisoblash jihatidan ancha og'irroq. Bu usulda keyingi nuqtadagi qiymatni aniqlash uchun bir nechta urinishlar amalga oshiriladi.

Biz bu maqolada ikkita usulni taqqoslaymiz eng soddasi va eng murakkabini va fizik jarayonlarga bog‘laymiz.

Eyler usuli — oddiy differensial tenglamalarni (ODT) taxminiy yechish uchun mo‘ljallangan eng sodda sonli usullardan biridir. U berilgan boshlang‘ich shart bilan birinchi tartibli ODTni yechish uchun ishlatiladi. Bu usuldan amalda kam foydalaniladi, chunki uning aniqligi past. Lekin oddiyligi sababli boshqa murakkabroq usullarni tushunish uchun asos bo‘lib xizmat qiladi.

Runge-Kutta usuli — bu oddiy differensial tenglamalar (ODT) uchun boshlang‘ich qiymat masalasini yechishda ishlatiladigan sonli usullar oilasidir. Runge-Kutta usuli ODT ning keyingi qiymatini topish uchun oraliqdagi bir nechta nuqtada funksiyaning egilishni (hosilasini) baholaydi. Eng keng tarqalgan va samarali variant bu **to‘rtinchi tartibli Runge-Kutta usuli (RK4)** hisoblanadi.

Eyler usuli formulasi:

Formulalar

ODT berilgan bo'lsin: $y' = f(x, y)$

Boshlang'ich shart: $y(x_0) = y_0$

Har bir qadamda yangi y qiymatini hisoblash uchun quyidagi formulalar qo'llaniladi:

$x_{i+1} = x_i + h$

$y_{i+1} = y_i + h \cdot f(x_i, y_i)$

Bu yerda:

- h – qadam o'lchami.
- i – qadam raqami (0 dan boshlanadi).

To'rtinchi tartibli Runge-Kutta (RK4) usuli formulasi :

$y' = f(x, y)$ ko'rinishidagi ODT uchun, boshlang'ich $y(x_0) = y_0$ sharti bilan, y_{n+1} ning taqribiy qiymati quyidagicha topiladi:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Bunda

h-qadam o'lchami

x_n, y_n -joriy nuqtalar.

$k_1 = hf(x_n, y_n)$, $k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$, $k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$, $k_4 = hf(x_n + h, y_n + k_3)$

RK4 usuli har bir qadamda to'rtta alohida egilish (qiymat) hisoblaydi:

1. k_1: Intervalning boshidagi egilish.
2. k_2: Intervalning o'rtasidagi egilish, k_1 yordamida hisoblangan.
3. k_3: Intervalning o'rtasidagi boshqa bir egilish, k_2 yordamida hisoblangan.
4. k_4: Intervalning oxiridagi egilish, k_3 yordamida hisoblangan.

Misol: ODTni Eyler usulida yechish.

$$y' = f(x, y) = 2y + x^2 - 1$$

Boshlang'ich shart: $y(0) = 1$
Maqsad: $h = 0.1$ qadam bilan $x = 0.2$ nuqtadagi y ning taxminiy qiymatini topish.

Eyler usulining formulasi quyidagicha:

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

Hisoblash bosqichlari

Biz ikki qadamda hisoblaymiz:

1. **Birinchi qadam:** $x_0 = 0$ dan $x_1 = 0.1$ gacha.
2. **Ikkinchi qadam:** $x_1 = 0.1$ dan $x_2 = 0.2$ gacha.

Birinchi qadam: $x_0 = 0$ dan $x_1 = 0.1$ gacha

Boshlang'ich qiymatlar: $x_0 = 0$, $y_0 = 1$, $h = 0.1$.
Berilgan funksiya: $f(x, y) = 2y + x^2 - 1$.

y_1 ni hisoblash ($x = 0.1$ nuqtasida):

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

$$y_1 = 1 + 0.1 \cdot (2 \cdot 1 + 0^2 - 1)$$

$$y_1 = 1 + 0.1 \cdot (1)$$

$$y_1 = 1.1$$

Demak, Eyler usuli bo'yicha $y(0.1) \approx 1.1$ ga teng.

Ikkinchi qadam: $x_1 = 0.1$ dan $x_2 = 0.2$ gacha

Endi boshlang'ich qiymatlarimiz: $x_1 = 0.1$, $y_1 = 1.1$.

y_2 ni hisoblash ($x = 0.2$ nuqtasida):

$$y_2 = y_1 + h \cdot f(x_1, y_1)$$

$$y_2 = 1.1 + 0.1 \cdot (2 \cdot 1.1 + (0.1)^2 - 1)$$

$$y_2 = 1.1 + 0.1 \cdot (2.2 + 0.01 - 1)$$

$$y_2 = 1.1 + 0.1 \cdot (1.21)$$

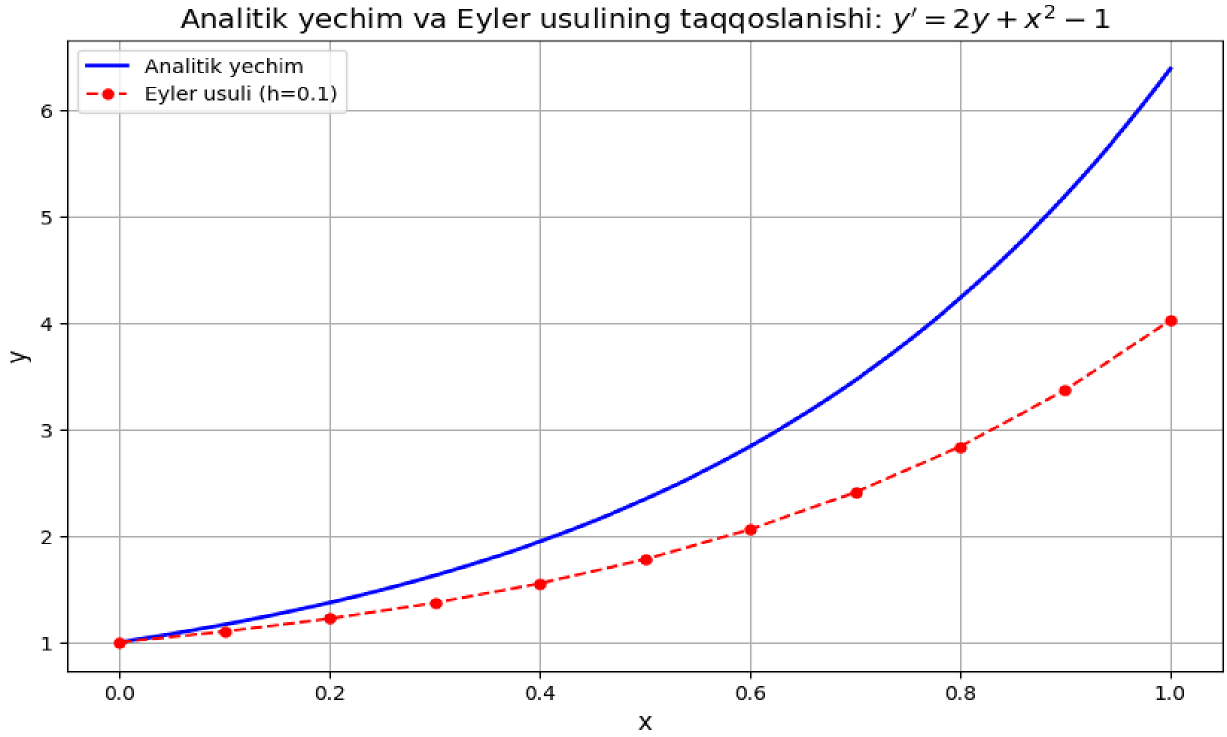
$$y_2 = 1.1 + 0.121$$

$$y_2 = 1.221$$

Xulosa.Eyler usulidan foydalanib, $f(x, y) = 2y + x^2 - 1$ differensial tenglamasining y(0)=1 boshlang'ich shartida x=0.2

nuqtasidagi taxminiy qiymati $y(0.2) \approx 1.221$ ga teng ekanligini aniqladi .

Endi buni analitik yechim bilan solishtiramiz



Bu usulda xatolik katta ekanini grafikdan ko'rishimiz mumkin.

Misol: ODTni Runge-Kutta usulida yechish.

Koshi masalasi berilgan bo'lsin:

$$y' = f(x, y) = 2y + x^2 - 1$$

Boshlang'ich shart: $y(0) = 1$

Maqsad: $h = 0.1$ qadam bilan $x = 0.2$ nuqtadagi y ning taxminiy qiymatini topish.

Biz ikki qadamda hisoblaymiz:

1. **Birinchi qadam:** $x_0 = 0$ dan $x_1 = 0.1$ gacha.
2. **Ikkinchi qadam:** $x_1 = 0.1$ dan $x_2 = 0.2$ gacha.

Birinchi qadam: $x_0 = 0$ dan $x_1 = 0.1$ gacha

Boshlang'ich qiymatlar: $x_0 = 0$, $y_0 = 1$, $h = 0.1$.
Berilgan funksiya: $f(x, y) = 2y + x^2 - 1$.

k_1 hisoblash:

$$k_1 = hf(x_0, y_0) = h(2y_0 + x_0^2 - 1)$$

$$k_1 = 0.1(2 \cdot 1 + 0^2 - 1) = 0.1(2 + 0 - 1) = 0.1 \cdot 1 = 0.1$$

k_2 hisoblash:

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$k_2 = 0.1\left(2(y_0 + \frac{k_1}{2}) + (x_0 + \frac{h}{2})^2 - 1\right)$$

$$k_2 = 0.1\left(2(1 + \frac{0.1}{2}) + (0 + \frac{0.1}{2})^2 - 1\right)$$

$$k_2 = 0.1\left(2(1.05) + (0.05)^2 - 1\right) = 0.1(2.1 + 0.0025 - 1) = 0.1(1.1025) = 0.11025$$

k_3 hisoblash:

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

$$k_3 = 0.1\left(2(y_0 + \frac{k_2}{2}) + (x_0 + \frac{h}{2})^2 - 1\right)$$

$$k_3 = 0.1\left(2(1 + \frac{0.11025}{2}) + (0.05)^2 - 1\right)$$

$$k_3 = 0.1\,(2(1.055125) + 0.0025 - 1) = 0.1(2.11025 + 0.0025 - 1) = 0.1(1.11275) = 0.111275$$

k_4 hisoblash:

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k_4 = 0.1\left(2(y_0 + k_3) + (x_0 + h)^2 - 1\right)$$

$$k_4 = 0.1\left(2(1 + 0.111275) + (0.1)^2 - 1\right)$$

$$k_4 = 0.1\,(2(1.111275) + 0.01 - 1) = 0.1(2.22255 + 0.01 - 1) = 0.1(1.23255) = 0.123255$$

y_1 ni topish ($x = 0.1$ nuqtasida):

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = 1 + \frac{1}{6}(0.1 + 2(0.11025) + 2(0.111275) + 0.123255)$$

$$y_1 = 1 + \frac{1}{6}(0.1 + 0.2205 + 0.22255 + 0.123255)$$

$$y_1 = 1 + \frac{1}{6}(0.666305) = 1 + 0.11105083$$

$$y_1 \approx 1.111051$$

Ikkinchi qadam: $x_1 = 0.1$ dan $x_2 = 0.2$ gacha

Endi boshlang'ich qiymatlarimiz: $x_1 = 0.1$, $y_1 = 1.111051$.

k_1 hisoblash:

$$k_1 = hf(x_1, y_1) = 0.1(2y_1 + x_1^2 - 1)$$

$$k_1 = 0.1(2(1.111051) + (0.1)^2 - 1) = 0.1(2.222102 + 0.01 - 1) = 0.1(1.232102) = 0.1232102$$

k_2 hisoblash:

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$$

$$k_2 = 0.1\left(2(1.111051 + \frac{0.1232102}{2}) + (0.1 + \frac{0.1}{2})^2 - 1\right)$$

$$k_2 = 0.1\left(2(1.1726561) + (0.15)^2 - 1\right) = 0.1(2.3453122 + 0.0225 - 1) = 0.1(1.3678122) = 0.13678122$$

k_3 hisoblash:

$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$$

$$k_3 = 0.1\left(2(1.111051 + \frac{0.13678122}{2}) + (0.15)^2 - 1\right)$$

$$k_3 = 0.1(2(1.17944161) + 0.0225 - 1) = 0.1(2.35888322 + 0.0225 - 1) = 0.1(1.38138322) = 0.138138322$$

k_4 hisoblash:

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$k_4 = 0.1\left(2(1.111051 + 0.138138322) + (0.2)^2 - 1\right)$$

$$k_4 = 0.1\,(2(1.249189322) + 0.04 - 1) = 0.1(2.498378644 + 0.04 - 1) = 0.1(1.538378644) = 0.1538378644$$

y_2 ni topish ($x = 0.2$ nuqtasida):

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_2 = 1.111051 + \frac{1}{6}(0.1232102 + 2(0.13678122) + 2(0.138138322) + 0.1538378644)$$

$$y_2 = 1.111051 + \frac{1}{6}(0.1232102 + 0.27356244 + 0.276276644 + 0.1538378644)$$

$$y_2 = 1.111051 + \frac{1}{6}(0.8268871484) = 1.111051 + 0.1378145247$$

$$y_2 \approx 1.248866$$

Xulosa

Runge-Kutta usulidan foydalanib, $y' = 2y + x^2 - 1$ differensial tenglamasining $y(0) = 1$ boshlang'ich shartidagi $x = 0.2$ nuqtasidagi taxminiy qiymati $\mathbf{y(0.2) \approx 1.248866}$ ga teng ekanligi aniqlandi. Bu natija usulning yuqori aniqligini ko'rsatadi.

Natija va xulosa

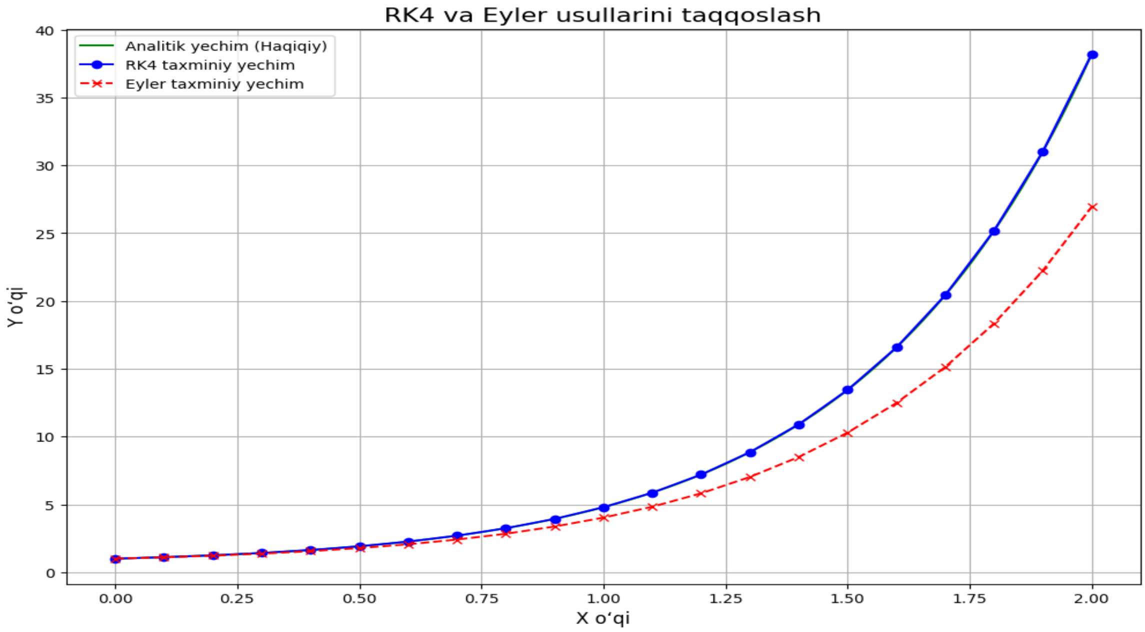
Eyler usulidan foydalanib, $y' = 2y + x^2 - 1$ differensial tenglamasining $y(0) = 1$ boshlang'ich shartidagi $x = 0.2$ nuqtasidagi taxminiy qiymati $\mathbf{y(0.2) \approx 1.221}$ ga teng ekanligi aniqlandi.

Runge-Kutta usulida esa bu qiymat $y(0.2) \approx 1.248866$ ga teng edi. Analitik yechimdan foydalanib, qaysi usul haqiqatga yaqinroq ekanligini osongina ko'rish mumkin:

- **Eyler usuli:** Taxminiy xatolik $1.248866 - 1.221 = 0.027866$.
- **RK4 usuli:** Taxminiy xatolik ancha kichikroq bo'ladi.

Bu solishtirish Eyler usulining oddiyroq, ammo aniqligi pastroq ekanligini, RK4 usulining esa murakkabroq, ammo aniqligi yuqori ekanligini isbotlaydi.

Bu ikkala usulni grafiklarini solishtirsak qaysi usul analitik yechimga yaxshi yaqinlashganini ko'ra olamiz.



Grafikdan ko'rinib turubdiki RK4 da deyarli hatolik ko'zga tashlanmaydi.

Runge-Kutta usuli real hayotdagi ko'plab fizik jarayonlarni modellashtirishda keng qo'llaniladi, chunki bu jarayonlarning aksariyati differensial tenglamalar bilan tasvirlanadi. Yuqorida ko'rilgan misolni fizik jarayonga bog'laymiz.

1. Elektr zanjiridagi o'zgaruvchan tok

Tasavvur qiling, sizda RLC zanjiri bor, bunda tokning o'zgarishi differensial tenglama bilan tasvirlanadi. Bu yerda y tok kuchi, x esa vaqtni ifodalaydi.

- **y' (tok kuchi o'zgarish tezligi):** Tok kuchining vaqtga nisbatan o'zgarish tezligi (kuchlanish manbai, qarshilik, induktivlik va sig'imga bog'liq).
- **$2y$ (qarshilik ta'siri):** Tok kuchining o'ziga bog'liq bo'lgan o'zgarish, bu qarshilikka bog'liq bo'lishi mumkin.
- **x^2-1 (tashqi kuchlanish manbai):** Vaqtga bog'liq ravishda o'zgaruvchi tashqi kuchlanish manbai. Misoldagi x^2-1 ko'rinishi kuchlanish manbai vaqt o'tishi bilan kvadratik bog'liqlikda o'zgarishini ko'rsatadi.

Boshlang'ich shart $y(0)=1$ esa $t=0$ vaqtida zanjirdagi tok kuchi 1 amperga teng bo'lganini bildiradi. Runge-Kutta usulidan foydalanib, biz istalgan vaqtda, masalan, $x=0.2$ soniyada zanjirdagi tokning taxminiy kuchini hisoblaymiz.

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