BOUSSINESQ TIPIDAGI KASR TARTIBLI TENGLAMALARI UCHUN KOSHI MASALASI

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Annotatsiya: Ushbu maqolada Boussinesq tipidagi vaqt bo‘yicha kasr tartibli subdiffuziya tenglamasi uchun Koshi masalasini o‘rganilgan va yechimining mavjudligi va yagonaligi shartlari aniqlangan.

Kalit so‘zlar: Boussinesq tipidagi kasr tenglama, Kaputo hosilasi, to‘g’ri masala, Fure usuli.

ON THE CAUCHY PROBLEM FOR A BOUSSINESQ TYPE TIME-FRACTIONAL SUBDIFFUSION EQUATIONS

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Abstract: In this paper, the Cauchy problem for a Boussinesq-type time-fractional subdiffusion equation is studied, and the conditions for the existence and uniqueness of its solution are determined.

Keywords: Boussinesq type equation, Caputo derivative, direct problem, Fourier method.

Kirish

Aytaylik, H separabel Hilbert fazosi bo‘lsin. H da sklayar ko‘paytmani $(\cdot,\cdot)$ kabi , normani esa $\|\cdot\|$ orqali belgilaymiz. $A:H\to H$ operator H Hilbert fazosida aniqlangan, o‘z-o‘ziga qo‘shma, musbat, chegaralanmagan ixtiyoriy operator bo‘lsin. Faraz qilaylik, A operator H fazoda to‘la ortonormal $\{v_k\}$ xos funksiyalar sistemasiga va ularga mos $\{\lambda_k\}$ musbat xos qiymatlar to‘plamiga ega bo‘lsin. Xos qiymatlarni qayta nomerlash orqali ularni kamaymaydigan qilib nomerlab olamiz, ya’ni, $0<\lambda_1\le\lambda_2\le\lambda_3\le\ldots$ ko‘rinishda yozib olamiz.

$\rho\in(1,2)$ berilgan son bo‘lsin. $C((a,b);H)$ orqali qiymatlari H da yotuvchi $t\in(a,b)$ oraliqda usluksiz $u(t)$ funksiyalar to‘plamini belgilaylik.

Faraz qilaylik $h(t)$ funksiya $[0,\infty)$ oraliqda aniqlangan bo‘lsin. $h(t)$ funksiyaning $1<\rho<2$ oraliqda Caputo ma’nosidagi kasr tartibli hosilasi quyidagi tenglik bilan aniqlanadi:

$$D_t^{\rho}h(t)=\frac{1}{\Gamma(2-\rho)}\int_0^t\frac{h''(\xi)}{(t-\xi)^{\rho-1}}d\xi\ ,\qquad t>0. \tag{1}$$

Masalaning qo‘yilishi va olingan asosiy natijalar

Quyidagi Koshi masalasini qaraymiz:

$$\begin{cases} D_t^{\rho}u(t)+D_t^{\rho}(Au(t))+Au(t)=0, & 0<t\le T;\\ u(+0)=\varphi, \\ u'(+0)=\psi, \end{cases} \tag{2}$$

bu yerda $\varphi,\psi\in H$, D_t^{ρ} – ρ chi tartibli Caputo ma’nosida kasr tartibli hosila.

Ushbu (2) masala Boussinesq tipidagi kasr tartibli hosilali tenglamalar uchun Koshi masalasi deyiladi.

Ta’rif. Agar $u(t)\in C((0,T];H)$ funksiya quyidagi $D_t^{\rho}u(t), D_t^{\rho}A(u(t)), Au(t)\in C((0,T];H)$ xossalarga ega bo‘lib, (2) masalaning barcha shartlarini qanoatlantirsa, u holda $u(t)$ funksiyaga (2) **Koshi masalasining yechimi** deyiladi.

(2) masalaning yechimini topish masalasiga ***to‘g’ri masala*** deyiladi.

Teorema. Aytaylik, $\varphi,\psi\in D(A)$ bo‘lsin. U holda (2) Koshi masalasining yagona yechimi mavjud va u quyidagi ko‘rinishga ega:

$$u(t)=\sum_{k=1}^{\infty}\left[\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)+\psi_k\, t\, E_{\rho,2}\left(-\mu_k t^{\rho}\right)\right]v_k. \tag{3}$$

bu yerda φ_k, ψ_k lar mos ravishda φ, ψ funksiyalarning Furye koeffitsiyentlari.

Dastlabki ma’lumotlar.

H da A operator quyidagicha aniqlanadi

$$Ah=\sum_{k=1}^{\infty}\lambda_k h_k v_k\ ,$$

bu yerda h_k lar $h\in H$ elementning Furye koeffisientlari. Ushbu operatorning aniqlanish sohasi quyidagi ko‘rinishga ega bo‘ladi:

$$D(A)=\left\{h\in H:\sum_{k=1}^{\infty}\lambda_k^2|h_k|^2<\infty\right\}.$$

$h\in D(A)$ elementning normasi esa quyidagi ko‘rinishda bo‘ladi:

$$\|h\|^2=\sum_{k=1}^{\infty}\lambda_k^2|h_k|^2=\|Ah\|^2\ .$$

Faraz qilaylik, $\rho>0$ va μ ixtiyoriy kompleks son bo‘lsin. $E_{\rho,\mu}(t)$ orqali ikki parametrli Mittag-Leffler funksiyasini belgilaymiz:

$$E_{\rho,\mu}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\rho n + \mu)} . \qquad (4)$$

Quyida biz Mittag-Leffler fuksiyasining ba'zi baholarini keltirib o'tamiz. Buning uchun quyidagi Wright funksiyasidan foydalanamiz:

$$\phi(\delta,\beta;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\beta+\delta k)} , \quad \delta > -1,\ \beta \in \mathbb{R},\ z \in \mathbb{C} .$$

Aytaylik, $1<\alpha<2$ bo'lsin. Professor A.V. Psxu ishlarida [qarang [3]] $t \geq 0$ da aniqlangan $h(t)$ funksiyalar uchun quyidagi integral almashtirishni aniqlab olamiz:

$$P^{\alpha,\beta}h(t) = t^{\beta-1}\int_0^{\infty} h(s)\,\phi\left(-\alpha,\beta;-\frac{s}{t^{\alpha}}\right)ds .$$

A.V. Psxu ishidan kelib chiquvchi quyidagi ba'zi lemmalarni keltirib o'tamiz.

Lemma 1. Aytaylik, $\gamma > 0$ bo'lsin . U holda

$$P^{\alpha,\beta}t^{\gamma-1} = t^{\alpha\gamma+\beta-1}\frac{\Gamma(\gamma)}{\Gamma(\alpha\gamma+\beta)}$$

tenglik o'rinli bo'ladi.

Lemma 1 dan Mittag-Leffler funksiyasining ta'rifi bo'yicha quyidagiga ega bo'lamiz:

$$P^{\xi,\eta}\left[t^{\mu-1}E_{\rho,\mu}(\lambda t^{\rho})\right] = t^{\mu\xi+\eta-1}E_{\rho\xi,\mu\xi+\eta}(\lambda t^{\rho\xi}) . \qquad (5)$$

Lemma 2. Faraz qilaylik, $1<\alpha<2$ bo'lsin. U holda quyidagi baho o'rinli:

$$\left|E_{\alpha,1}(-t^{\alpha})\right| < 1, \quad t > 0 .$$

Lemma 3. Aytaylik, $1<\alpha<2$ bo'lsin. U holda quyidagi baho o'rinli:

$$\left|E_{\alpha,2}(-t^{\alpha})\right| < 1, \quad t > 0 .$$

To'g'ri masala

Ushbu bo'limda biz (2) to'g'ri masala ni yechishga e'tibor qaratamiz.

Isbot. Yechimning mavjudligi. (2) masalani yechish uchun matematik fizika masalalrini yechishda eng keng qo'llaniladigan usullaridan biri Furye usulidan foydalanamiz. Buning uchun tenglamaning yechimini quyidagi ko'rinishda izlaymiz:

$$u(t) = \sum_{k=1}^{\infty} T_k(t)\, v_k . \qquad (6)$$

(6) qatorni (2) masalaning 1-tenglamasiga qo'yib, ikkinchi va uchinchi shatrlardan foydalansak, quyidagi Koshi masalasiga kelamiz:

$$\begin{cases} D_t^{\rho}T_k(t) + \mu_k T_k(t) = 0, \\ T_k(+0) = \varphi_k, \\ T_k'(+0) = \psi_k. \end{cases} \qquad (7)$$

(7) masalaning yechimi quyidagi ko'rinishda bo'ladi (qarang [1]):

$$T_k(t) = \varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right) + \psi_k t\, E_{\rho,2}\left(-\mu_k t^{\rho}\right)$$

Agar ushbu ifodani (6)-qatorga qo'ysak, unda formal yechim (3) ni hosil qilamiz. Ushbu yechimning haqiqiy yechim ekanligini ko'rsatish uchun, $u(t)$ funksiya **ta'rifdagi barcha shartlarni** qanoatlantirishini isbotlash zarur. Avvalo, (3)-qatorning **tekis yaqinlashuvchi** ekanligini ko'rsatamiz. (3)-qatorning qisman yig'indisini quyidagicha belgilaymiz:

$$S_n(t) = \sum_{k=1}^{n}\left[\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right) + \psi_k\, t\, E_{\rho,2}\left(-\mu_k t^{\rho}\right)\right] v_k ,$$

Parseval tengligiga ko'ra

$$\sum_{k=1}^{n}\left|\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right) + \psi_k\, t\, E_{\rho,2}\left(-\mu_2 t^{\rho}\right)\right|^2 \le$$

$$\le C\sum_{k=1}^{n}\left|\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right|^2 + C\sum_{k=1}^{n}\left|\psi_k\, t\, E_{\rho,2}\left(-\mu_k t^{\rho}\right)\right|^2 = S_n^1 + S_n^2 ,$$

bu yerda

$$S_n^1 = C\sum_{k=1}^{n}\left|\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right|^2, \qquad S_n^2 = C\sum_{k=1}^{n}\left|\psi_k t\, E_{\rho,2}\left(-\mu_k t^{\rho}\right)\right|^2 .$$

S_n^1 va S_n^2 ni baholash uchun mos ravishda **Lemma 2** va **Lemma 3 dan** foydalanamiz. Shunda:

$$S_n^1 = C\sum_{k=1}^{n}\left|\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right|^2 \le C\sum_{k=1}^{n}\left|\varphi_k\right|^2$$

$$S_n^2 = C\sum_{k=1}^{n}\left|\psi_k\, t\, E_{\rho,2}\left(-\mu_k t^{\rho}\right)\right|^2 \le C\, t^2\sum_{k=1}^{n}\left|\psi_k\right|^2$$

Demak, agar $\varphi, \psi \in H$ bo'lsa, u holda (3) qator yaqinlashuvchi bo'ladi. Bundan $u(t) \in C((0,T];H)$ ekanligi kelib chiqadi.

Endi $Au(t)$ ni baholaylik. A opertorning ta'rifiga va Parseval tengligiga ko'ra

$$\left\|AS_n(t)\right\|^2 = \left\|\sum_{k=1}^{n}\left[\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right) + \psi_k\, t\, E_{\rho,2}\left(-\mu_k t^{\rho}\right)\right]\lambda_k v_k\right\|^2 =$$

$$= \sum_{k=1}^{n}\lambda_k^2\left|\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right) + \psi_k\, t\, E_{\rho,2}\left(-\mu_k t^{\rho}\right)\right|^2$$

$$\left\|AS_n(t)\right\|^2 = \sum_{k=1}^{n}\lambda_k^2\left|\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right) + \psi_k\, t\, E_{\rho,2}\left(-\mu_2 t^{\rho}\right)\right|^2 \le$$

$$\le C\sum_{k=1}^{n}\lambda_k^2\left|\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right|^2 + C\sum_{k=1}^{n}\lambda_k^2\left|\psi_k\, t\, E_{\rho,2}\left(-\mu_k t^{\rho}\right)\right|^2 = AS_n^1 + AS_n^2$$

bu yerda

$$AS_n^1 = C\sum_{k=1}^{n} \lambda_k^2 \left|\varphi_k E_{\rho,1}\left(-\mu_k t^\rho\right)\right|^2, \qquad AS_n^2 = C\sum_{k=1}^{n} \lambda_k^2 \left|\psi_k\, t\, E_{\rho,2}\left(-\mu_k t^\rho\right)\right|^2.$$

U holda quyidagi tengsizlikka ega bo'lamiz:

$$\left\|AS_n(t)\right\|^2 \le AS_n^1 + AS_n^2.$$

Endi, **Mittag-Leffler funksiyasining bahosidan** foydalanib, quyidagi **bahoni** hosil qilamiz.

$$AS_n^1 = C\sum_{k=1}^{n} \lambda_k^2 \left|\varphi_k E_{\rho,1}\left(-\mu_k t^\rho\right)\right|^2 \le C\sum_{k=1}^{n} \lambda_k^2 \left|\varphi_k\right|^2,$$

$$AS_n^2 = C\sum_{k=1}^{n} \lambda_k^2 \left|\psi_k\, t\, E_{\rho,2}\left(-\mu_k t^\rho\right)\right|^2 \le C\sum_{k=1}^{n} \lambda_k^2 \left|\psi_k\, t\right|^2 \le C\, t^2 \sum_{k=1}^{n} \lambda_k^2 \left|\psi_k\right|^2$$

Ko'rinib turibdiki, agar $\varphi, \psi \in D(A)$ bo'lsa u holda $Au(t) \in C((0,T];H)$ bo'ladi.

Endi esa $A\left(D_t^\rho u(t)\right)$ ni baholaylik. Quyidagi $D_t^\rho S_n(t)$ yig'indini $t \in (0,T]$ da tekis yaqinlashishini ko'rsatamiz:

$$(I+A)^{-1} AS_n(t) = \sum_{k=1}^{n} \frac{\lambda_k}{1+\lambda_k}\left[\varphi_k E_{\rho,1}\left(-\mu_k t^\rho\right) + \psi_k\, t\, E_{\rho,2}\left(-\mu_k t^\rho\right)\right] v_k.$$

Parseval tengligiga ko'ra

$$\left\|(I+A)^{-1} AS_n(t)\right\|^2 = \sum_{k=1}^{n} \frac{\lambda_k^{\;2}}{\left(1+\lambda_k\right)^2}\left|\varphi_k E_{\rho,1}\left(-\mu_k t^\rho\right) + \psi_k\, t\, E_{\rho,2}\left(-\mu_k t^\rho\right)\right|^2 \le$$

$$\le C\sum_{k=1}^{n} \frac{\lambda_k^2}{\left(1+\lambda_k\right)^2}\left|\varphi_k E_{\rho,1}\left(-\mu_k t^\rho\right)\right|^2 + C\sum_{k=1}^{n} \frac{\lambda_k^2}{\left(1+\lambda_k\right)^2}\left|\psi_k\, t\, E_{\rho,2}\left(-\mu_k t^\rho\right)\right|^2$$

Mittag-Leffler funksiyasining Lemma 2 va Lemma 3 dagi bahosidan foydalanib va $\dfrac{\lambda_k}{1+\lambda_k} \le 1$ ni hisobga olsak quyidagi bahoga ega bo'lamiz:

$$\left\|(I+A)^{-1} AS_n(t)\right\|^2 \le C_1 \sum_{k=1}^{n} \left|\varphi_k\right|^2 + C_2 t^2 \sum_{k=1}^{n} \left|\psi_k\right|^2$$

bu yerda C_1, C_2 o'zgarmas musbat sonlar. Bundan chiqadi, agar $\varphi,\ \psi \in H$ bo'lsa, u holda $D_t^\rho u(t) \in C((0,T];H)$ bo'ladi. Agar $\varphi, \psi \in D(A)$ bo'lsa $Au(t), D_t^\rho u(t) \in C((0,T];H)$ ekanligi va $A\left(D_t^\rho u(t)\right) = -Au(t) - D_t^\rho u(t)$ tenglikdan $\varphi, \psi \in D(A)$ bo'lsa $A\left(D_t^\rho u(t)\right)) \in C((0,T];H)$ ekanligi kelib chiqadi.

Endi esa (2) dagi ikkinchi shartni tekshiramiz.

$$u'(t) = \frac{d}{dt}\sum_{k=1}^{\infty}\left[\varphi_k E_{\rho,1}\left(-\mu_k t^\rho\right) + \psi_k\, t\, E_{\rho,2}\left(-\mu_k t^\rho\right)\right] v_k =$$

$$=\sum_{k=0}^{\infty}\left[\varphi_k\left(-\mu_k\right)t^{\rho-1}E_{\rho,\rho}\left(-\mu_k t^{\rho}\right)+\left(\psi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right)\right]v_k\,.$$

$t=0$ bo'lganda va $1<\rho<2$ ni hisobga olsak $u'(0)=\psi_k$ ekanligi kelib chiqadi.

Yechimning yagonaligi. (2) masalaning yechimi yagonaligini isbotlash uchun standart usuldan foydalanamiz, ya'ni

$$\begin{cases} D_t^{\rho}u(t)+D_t^{\rho}(Au(t))+Au(t)=0, & 0<t\le T; \\ u(+0)=0, \\ u'(+0)=0, \end{cases} \tag{8}$$

masala aynan nol yechimga ega ekanligini ko'rsatish yetarli. Aytaylik, $u(t)$ ixtiyoriy yechim bo'lib, u holda $u_k(t)=\left(u(t),v_k\right)$ bo'lsin. $u(t)=\sum_{k=1}^{\infty}u_k(t)\,v_k$ ekanligi va A operatorning o'z-o'ziga qo'shmaligidan, (8) ga ko'ra quyidagi

$$D_t^{\rho}u(t)=-\left(D_t^{\rho}A(u(t))+A(u(t))\right)$$

tenglikdan

$$D_t^{\rho}u_k(t)=\left(D_t^{\rho}u(t),v_k\right)=-\left(A\left(D_t^{\rho}u(t)\right)+Au(t),v_k\right)=$$

$$=-\left(A\left(D_t^{\rho}u(t)\right),v_k\right)-\left(Au(t),v_k\right)=-\left(D_t^{\rho}u(t),Av_k\right)-\left(u(t),Av_k\right)=$$

$$=-\left(D_t^{\rho}u(t),\lambda_k v_k\right)-\left(u(t),\lambda_k v_k\right)=-\lambda_k D_t^{\rho}u_k(t)-\lambda_k u_k(t)$$

tenglikka ega bo'lamiz. Bundan quyidagi Koshi masalasiga kelamiz:

$$\begin{cases} D_t^{\rho}u_k(t)+\dfrac{\lambda_k}{1+\lambda_k}u_k(t)=0, \\ u_k(0+)=0, \\ u_k'(0+)=0. \end{cases}$$

Bu masalani yechib (qarang [1]), barcha $k\ge 1$ lar uchun $u_k(t)=0$ ekanligini topamiz. $\{v_k\}$ xos funksiyalarning to'laligidan $u(t)\equiv 0$ ekanligi kelib chiqadi. Teorema isbotlandi.

FOYDALANILGAN ADABIYOTLAR RO'YXATI

1. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier (2006).

2. R. Ashurov, Yu. Fayziev, "On the nolocal boundary value problems for time-fractional equations," Fractal and Fractional, 6, 41 (2022).

3. A.V.Pskhu The Stankovich Integral transform and Its Applications, Special Functions and Analysis of Differntial Equations, 2020, p.16.

4. Umarov S., Hahn M., Kobayashi K. Beyond the triangle: Browian motion, Ito calculas, and Fokker-Plank equation-fractional generalizations. World Scientific. 2017.

5. V.S. Vladimirov STEKLOV INSTITUTE OF MATHEMATICS MOSCOW, U.S.S.R. Equations of Mathematical Physics, MARCEL DEKKER, New York 1971

6. Salohiddinov "Matematik fizika tenglamalari" O'zbekiston" 2002. В.С.Владимиров, В.В.Жаринов "Уравнения математической физики" - Москва: ФИЗМАТ, 2004.

7. Zhang Y., Benson D.A., Meerschaert M.M., LaBolle E.M., Scheffler H.P. Random walk approximation of fractional-order multiscaling anomalus deffusion. //Phys. Rev. E. 2006. V. 74.

8. Ashurov R., Fayziev Yu., Nosirova D., Amrullaeva D., Latipova Sh. On the non-local problems for a Boussinesq type time-fractional subdiffusion equations // Amaliy matematika va axborot texnologiyalarninig zamonaviy muammolari, 11-12 may, 2022 йил, Buxoro, b. 143.