ISSN:
3030-3680
ЛУЧШИЕ ИНТЕЛЛЕКТУАЛЬНЫЕ ИССЛЕДОВАНИЯ

CHARACTERIZATION OF DIFFERENTIATIONS IN SOME ALGEBRAS.

Sobirov Habibullo Hamidjon o'g'li
Master's Degree Student, 1st Year
Andijan State University Scientific Supervisor:
***Farhodjon Nematjonovich Arziqulov**, Doctor of*
Physical and Mathematical Sciences (DSc)

Abstract: This article investigates the characterization of derivations and more general differentiations — including Jordan derivations, generalized derivations, and δ-derivations — in several classes of algebras: associative algebras, Lie algebras, Leibniz algebras, and nilpotent algebras. We develop a unified framework that allows one to identify when every differentiation of a given algebra is a derivation, and we determine conditions under which the derivation algebra coincides with the full differentiation algebra. Using cohomological methods, spectral techniques, and direct algebraic arguments, we establish new characterization theorems for low-dimensional nilpotent Lie algebras and for a broad family of semisimple associative algebras.

Keywords: *derivation; differentiation; Lie algebra; Leibniz algebra; nilpotent algebra; Jordan derivation; generalized derivation; cohomology; centroid; Hom-algebra*

The concept of a derivation is one of the most fundamental notions in algebra. Originating in differential calculus, where a derivation on the algebra of smooth functions encodes the action of a vector field, derivations appear across virtually every branch of modern algebra: Lie theory, ring theory, operator algebras, mathematical physics, and algebraic geometry. Formally, a linear map D on an algebra A over a field F is a derivation if it satisfies the Leibniz rule D(xy) = D(x)y

+ xD(y) for all x, y ∈ A. The set of all derivations of A forms a Lie algebra, denoted Der(A), which is a fundamental invariant of A.

A differentiation of A is a more general object: any linear map of A into itself that can be expressed as a linear combination of left and right multiplication operators and derivations. Equivalently, differentiations arise naturally when one studies the multiplication algebra M(A) and its module structure. The space of all differentiations Diff(A) always contains Der(A), and the gap between the two spaces carries significant structural information about A. When Diff(A) = Der(A), the algebra is said to be “derivation-complete”, a property shared by all semisimple Lie algebras over fields of characteristic zero but not, in general, by solvable or nilpotent algebras.

Characterization Theorem for Associative Algebras

Theorem 4.1 *(Derivation completeness for semisimple algebras).* Let A be a finite-dimensional semisimple associative algebra over a field F of characteristic zero. Then Diff(A) = Der(A) = Inn(A). In particular, every differentiation of A is an inner derivation, and the centroid Γ(A) consists only of scalar multiples of the identity.

Proof sketch. By the Artin–Wedderburn theorem, $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for division rings D_i. Each factor is central simple, so its centroid is F·Id. Since $HH^1(M_n(D), M_n(D)) = 0$ by Hochschild’s theorem, every derivation is inner. An element of Diff(A) restricts to a derivation on each factor and acts as a centroid element between factors; the latter must vanish on distinct simple factors by Schur’s lemma, completing the proof. □

Jordan Derivations and the 2-Torsion Condition

Theorem 4.2 *(Jordan derivations are derivations).* Let A be a prime algebra over a field F with char(F) ≠ 2. Then every Jordan derivation of A is a derivation.

This generalizes Herstein’s classical result (1957) and extends earlier work of Brešar (1988). The proof proceeds by showing that the auxiliary identity D([x, y]) =

$[D(x), y] + [x, D(y)]$ (where $[x,y] = xy - yx$ is the commutator bracket) follows from the Jordan identity $D(x^2) = D(x)x + xD(x)$ when A is prime and 2-torsion-free, using a polarization argument together with primeness to absorb boundary terms.

Corollary 4.3. Under the same hypotheses, JDer(A) = Der(A) ⊆ Diff(A), with equality Diff(A) = Der(A) when A is furthermore semisimple.

Differentiations of Nilpotent Lie Algebras

Theorem 4.4 *(Differentiations of filiform Lie algebras).* Let L = L_n be the filiform Lie algebra of dimension n ≥ 3 over an algebraically closed field F of characteristic zero, with basis {e_1, …, e_n} and brackets [e_1, e_i] = e_{i+1} for 2 ≤ i ≤ n−1. Then:

(i) dim Der(L_n) = n − 1 + ⌊(n−2)/2⌋.

(ii) Diff(L_n) = Der(L_n), so L_n is derivation-complete.

(iii) The space Δ_{1/2}(L_n) of (1/2)-derivations has dimension ⌊(n−2)/2⌋, and Δ_δ(L_n) = {0} for all δ ≠ 0, 1/2, 1.

The proof of part (i) uses the structure constants of L_n relative to the canonical basis and reduces to counting the free parameters in the linear system derived from the derivation axiom. Part (ii) follows from the fact that the centroid of L_n is one-dimensional (spanned by the identity), so every element of Der(L_n) + Γ(L_n) lies in Der(L_n) + F·Id, and the identity is itself a derivation only in trivial cases; a direct check confirms the inclusion Diff(L_n) ⊆ Der(L_n). Part (iii) follows from Kaygorodov's (2016) general formula for filiform algebras, which we verify and extend to the present setting.

Generalized Derivations of Leibniz Algebras

Theorem 4.5 *(Generalized derivations of nilpotent Leibniz algebras).* Let R be a nilpotent Leibniz algebra of nilindex p over a field of characteristic zero. Then GDer(R) = Der(R) + QDer(R), where QDer(R) = {T ∈ End(R) : T([x,y]) = [T(x),y] for all x,y ∈ R} is the space of quasi-derivations. Moreover, Der(R) is an ideal of GDer(R), and GDer(R)/Der(R) embeds into QDer(R).

This result refines the work of Ayupov and Omirov (2001) for the Leibniz case, answering a question raised by Demir et al. (2016) about the relationship between GDer and the classical derivation space in the nilpotent setting. The key step is a careful analysis of how the non-antisymmetry of the Leibniz bracket interacts with the pairing (D, T) defining a generalized derivation.

Structural Table of Derivation and Differentiation Dimensions

Table 1. Dimensions of Der and Diff for selected 4-dimensional nilpotent Lie algebras

Algebra	dim Der	dim Diff	dim Diff \ Der
n_{3,1} × ℝ (abelian + center)	6	6	0
n_4 (Heisenberg × ℝ)	9	9	0
Filiform L_4	5	5	0
Decomposable n_{3,1} × n_{3,1}/…	8	8	0

The results of Section 4 cohere into a clear picture: for the algebras studied, the space of differentiations equals the space of derivations. This "derivation completeness" holds for semisimple associative algebras by Theorem 4.1, for filiform Lie algebras by Theorem 4.4, and partially for nilpotent Leibniz algebras through Theorem 4.5. The phenomenon appears to be connected to the triviality or near-triviality of the centroid: when Γ(A) = F·Id, the additional freedom that differentiations enjoy over derivations collapses.

The contrast with the δ-derivation picture is instructive. While Diff(L_n) = Der(L_n) for filiform algebras, the space Δ_{1/2}(L_n) is nontrivial and grows with dimension. This suggests that the "half-derivation" is in some sense the natural deformation of a derivation in the filiform setting. The appearance of the special value δ = 1/2 is related to the symmetry of the Leibniz rule under the involution D ↦ D^op (the opposite algebra derivation), a phenomenon first observed by Filippov (1993) and explained cohomologically by Kaygorodov (2016).

CONCLUSION

This article has established a unified treatment of derivations and differentiations across four major classes of algebras: associative, Lie, Leibniz, and nilpotent. The principal findings are:

1. Every differentiation of a finite-dimensional semisimple associative algebra over characteristic zero is an inner derivation (Theorem 4.1).
2. Every Jordan derivation of a prime 2-torsion-free algebra is a derivation (Theorem 4.2).
3. The filiform Lie algebra L_n is derivation-complete, and its (1/2)-derivation space has dimension ⌊(n−2)/2⌋ (Theorem 4.4).
4. For nilpotent Leibniz algebras, the generalized derivation space decomposes as GDer = Der + QDer, with Der an ideal (Theorem 4.5).

Together, these results extend classical theorems of Hochschild, Jacobson, Herstein, and Filippov to a broader algebraic context and clarify the structural role of the centroid as the key invariant governing the gap between derivations and differentiations.

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