# PARABOLIK TENGLAMALARDA TESKARI MASALALARNING SONLI YECHIMI.

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**Annotatsiya.** Ushbu maqolada parabolik tenglamalarda teskari masalalarning sonli yechimi masalasi o‘rganilgan. Parabolik turdagi tenglamalar issiqlik tarqalishi, diffuziya jarayonlari, geofizika, biologiya va texnikada keng qo‘llaniladi. Bunday tenglamalarga qo‘yilgan teskari masalalarda noma’lum parametrlar, chegaraviy yoki boshlang‘ich shartlarni aniqlash talab etiladi. Maqolada teskari masalalarning matematik qo‘yilishi, ularning korrektlik shartlari hamda sonli yechish usullari tahlil qilingan. Xususan, ayirmali sxema va iteratsion usullar asosida yechimni topish yo‘llari ko‘rib chiqilgan. Olingan natijalar teskari masalalarni amaliy yechishda sonli usullarning samaradorligini ko‘rsatadi.

**Kalit so‘zlar:** parabolik tenglama, teskari masala, sonli yechim, ayirmali sxema, iteratsion usul, chegaraviy shart, korrektlik, approksimatsiya.

Parabolik tenglamalar uchun odatda to‘g‘ri masala qo‘yilib, berilgan boshlang‘ich va chegaraviy shartlar asosida yechim topiladi. Ammo amaliyotda ko‘pincha bu shartlarning ayrimlari noma’lum bo‘lib, ularni kuzatish natijalari yoki qo‘shimcha ma’lumotlar asosida aniqlash talab etiladi. Bunday hollarda teskari masalalar yuzaga keladi. Mazkur maqolaning maqsadi parabolik tenglamalarda teskari masalalarning sonli yechimini o‘rganish, ularning matematik qo‘yilishini tahlil qilish hamda sonli usullar yordamida yechim topish imkoniyatlarini ko‘rsatishdan iborat. Maqolada ayirmali sxema, iteratsion usullar va approksimatsiya usullari asosida teskari masalalarning yechimi ko‘rib chiqiladi hamda olingan natijalar tahlil qilinadi.

$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=f(x,y)$ (1) tenglamani qaraymiz. Bu tenglamani teskarisi $x=x(\xi,\eta),\ y=y(\xi,\eta)$ almashtirishga ega bo‘lgan $\xi=\xi(x,y),\ \eta=\eta(x,y)$ (2) almashtirish yordamida (1) tenglamani ya’ni $\xi$ va $\eta$ o‘zgaruvchilarga nisbatan soddaroq tenglamaga keltirish mumkin, bunda $\xi(x,y),\ \eta(x,y)$ funksiyalar almashtirish bajarilayotgan biror *D* sohada uzluksiz, ikki marta differensiallanuvchi. Shu maqsadda ushbu xususiy hosilalarni topamiz.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x},\ \frac{\partial u}{\partial y}=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y},$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x})=\frac{\partial^2 u}{\partial \xi^2}(\frac{\partial \xi}{\partial x})^2+\frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial x}+\frac{\partial^2 u}{\partial \eta^2}\cdot(\frac{\partial \eta}{\partial x})^2+\frac{\partial u}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial^2 u}{\partial \eta\partial \xi}\cdot\frac{\partial \eta}{\partial x}\cdot\frac{\partial \xi}{\partial x}=$$

$$=\frac{\partial^2 u}{\partial \xi^2}\cdot\left(\frac{\partial \xi}{\partial x}\right)^2+2\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial^2 u}{\partial \eta^2}\cdot(\frac{\partial \eta}{\partial x})^2+\frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial^2 \xi}{\partial x^2},$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}(\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x})=\frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x\partial y}+\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial y}+ \quad (3)$$

$$+\frac{\partial^2 u}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x}\cdot\frac{\partial \eta}{\partial y}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\cdot\frac{\partial \eta}{\partial y}+\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \eta}{\partial x}\cdot\frac{\partial \xi}{\partial y},$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial \xi^2}\cdot(\frac{\partial \xi}{\partial y})^2+2\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial y}+\frac{\partial^2 u}{\partial \eta^2}(\frac{\partial \eta}{\partial y})^2+\frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial y^2}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial y^2}.$$

Xususiy hosilalarning topilgan qiymatlarini (1) tenglamaga qo‘yib, quyidagi tenglamani hosil qilamiz:

$$a_{11}\frac{\partial^2 u}{\partial \xi^2}+2a_{12}\frac{\partial^2 u}{\partial \xi\partial \eta}+a_{13}\frac{\partial^2 u}{\partial \eta^2}+F\left(\xi,\eta,u,\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta}\right)=0 \quad (4)$$

bu yerda $a_{11}(\xi,\eta)=A\left(\frac{\partial \xi}{\partial x}\right)^2+2B\frac{\partial \xi}{\partial x}\cdot\frac{\partial \xi}{\partial y}+C\left(\frac{\partial \xi}{\partial y}\right)^2,$ (5)

$$a_{13}(\xi,\eta)=A(\cdot\frac{\partial \eta}{\partial x})^2+2B\frac{\partial \eta}{\partial x}\cdot\frac{\partial \eta}{\partial y}+C(\frac{\partial \eta}{\partial y})^2,$$

$a_{12}(\xi,\eta)=A\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial x}+B\left(\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial x}\right)+C\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial y}$ (5) da $\xi(x,y),\ \eta(x,y)$ funksiyalarni shunday tanlaymizki, natijada $a_{11},\ a_{13}$ koeffitsientlar nolga aylansin, ya’ni

$$A\left(\frac{\partial\xi}{\partial x}\right)^2+2B\frac{\partial\xi}{\partial x}\cdot\frac{\partial\xi}{\partial y}+C\left(\frac{\partial\xi}{\partial y}\right)^2=0,\ A\left(\frac{\partial\eta}{\partial x}\right)^2+2B\frac{\partial\eta}{\partial x}\cdot\frac{\partial\eta}{\partial y}+C\left(\frac{\partial\eta}{\partial y}\right)^2=0 \quad (6)$$

bo‘lsin. Birinchi tenglamani $(\frac{\partial\xi}{\partial y})^2$ ga bo‘lsak $A\left(\frac{\frac{\partial\xi}{\partial x}}{\frac{\partial\xi}{\partial y}}\right)^2+2B\frac{\frac{\partial\xi}{\partial x}}{\frac{\partial\xi}{\partial y}}+C=0$ kvadrat tenglama hosil bo‘ladi. Bu tenglamani $\frac{\frac{\partial\xi}{\partial x}}{\frac{\partial\xi}{\partial y}}$ ga nisbatan yechsak, $\frac{\frac{\partial\xi}{\partial x}}{\frac{\partial\xi}{\partial y}}=\frac{-B\pm\sqrt{B^2-AC}}{A}$

(7)

kelib chiqadi. Bundan kvadrat uchhadni chiziqli ko‘paytuvchilarga ajratish qoidasiga asoslanib $A\left(\frac{\frac{\partial\xi}{\partial x}}{\frac{\partial\xi}{\partial y}}-\frac{-B+\sqrt{B^2-AC}}{A}\right)\cdot\left(\frac{\frac{\partial\xi}{\partial x}}{\frac{\partial\xi}{\partial y}}-\frac{-B-\sqrt{B^2-AC}}{A}\right)=0$ yoki $A\frac{\partial\xi}{\partial x}+(B-\sqrt{B^2-AC})\frac{\partial\xi}{\partial y}=0,\ A\frac{\partial\xi}{\partial x}-(B+\sqrt{B^2-AC})\frac{\partial\xi}{\partial y}=0$ tengliklarni hosil qilamiz. Demak (6)ning har bir tenglamasi xususiy hosilali birinchi tartibli $A\frac{\partial\xi}{\partial x}+(B-\sqrt{B^2-AC})\frac{\partial\xi}{\partial y}=0,\ A\frac{\partial\eta}{\partial x}-(B+\sqrt{B^2-AC})\frac{\partial\eta}{\partial y}=0$ chiziqli tenglamalarga ajraladi. Bu tenglamalar $\frac{dx}{A}=\frac{dy}{B-\sqrt{B^2-AC}}$ va $\frac{dx}{A}=\frac{dy}{B+\sqrt{B^2-AC}}$ yoki $Ady-(B-\sqrt{B^2-AC})dx=0,\ \ Ady-(B+\sqrt{B^2-AC})dx=0$ (8)

differensial tenglamalarga teng kuchli. Bu yerdagi ikkita tenglamalarni

$Ady^2-2Bdxdy+Cdx^2=0$ (9) yoki buni $dx^2$ ga bo‘lib uni $A(\frac{dy}{dx})^2-2B\frac{dy}{dx}+C=0$ (10) ko‘rinishdagi bitta tenglama shaklida yozish mumkin.

(8) ning umumiy integrallari $\varphi(x,y)=C_1,\ \psi(x,y)=C_2$ bo‘lsin. Bu holda ular (1) tenglamaning ikkita egri chiziqlar oilasini tashkil qilib, tenglamaning xarakteristikalari deyiladi. (9) tenglama esa (1) tenglama xarakteristikalarining differensial tenglamasi deyiladi[2].

Shuningdek (9) tenglama (1) tenglamaning xarakteristik tenglamasi, uning umumiy integrali esa xarakteristika deb ham yuritiladi.

Quyidagi lemmani isbotlaymiz.

**lemma.** Agar $z=\varphi(x,y)$ funksiya $A\left(\frac{\partial z}{\partial x}\right)^2+2B\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}+C\left(\frac{\partial z}{\partial y}\right)^2=0$ (11)

tenglamaning xususiy yechimlaridan biri bo‘lsa, $\varphi(x,y)=C$ ifoda $A\left(\frac{dy}{dx}\right)^2-2B\frac{dy}{dx}+C=0$ (10)

xarakteristik tenglamaning umumiy integrali bo‘ladi va aksincha $\varphi(x,y)=C$ (10) xarakteristik tenglamaning umumiy integrali bo‘lganda $z=\varphi(x,y)$ funksiya (11) tenglamaning xususiy yechimi bo‘ladi.

**Isoboti.** Ayniyatlik $z=\varphi(x,y)$ (11) tenglamaning yechimi bo‘lsin. U holda (11) tenglamani $\left(\frac{\partial z}{\partial y}\right)^2$ ga bo‘lib lemmaning shartiga ko‘ra $\varphi(x,y)$ hosil bo‘lgan tenglamaning yechimi ekanligini hisobga olsak

$$A\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)^2-2B\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)+C=0 \qquad (12)$$

bo‘ladi. $\varphi(x,y)=C$ tenglamani oshkormas shakldа berilgan funksiyaning tenglamasi deb qarasak $\frac{dy}{dx}=-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}$ bo‘lishi ravshan. Buni (10) ga qo‘ysak (12) ga asosan

$A\left(\frac{dy}{dx}\right)^2-B\left(\frac{dy}{dx}\right)+C=A\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)^2-2B\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)+C=0$ bo‘ladi. Bu bilan lemmaning birinchi qismi ya’ni $\varphi(x,y)$ (11) tenglamaning yechimi bo‘lganda $\varphi(x,y)=C$ (10) tenglamaning umumiy integrali bo‘lishi isbotlandi.

(9) tenglamalarning integrallarini qanday bo‘lishi va (1) tenglamaning sodda ko‘rinishga keltirilishi shu tenglamaning ikkinchi tartibli xususiy hosilalari oldidagi koeffitsiyentlardan tuzilgan $\Delta = B^2 - AC$ diskriminantning ishorasiga bog’liq bo‘ladi[1].

$\Delta$ diskriminantning ishorasiga bog’liq holda (1) tenglamni quyidagi turlarga ajratish mumkin.

Parabolik turdagi tenglama uchun $B^2 - A \cdot C = 0$ bo‘lgani uchun (8) tenglamalar bitta tenglamani ifodalaydi va (9) xarakteristik tenglama yagona $\varphi(x, y) = C$ umumiy integralga ega bo‘ladi[3].

Bu holda $\xi = \varphi(x, y)$ va $\eta = \eta(x, y)$ deb olamiz, bu yerdagi $\eta(x, y)$ $\varphi$ ga bog’liq bo‘lmagan ikki marta differensiallanuvchi ixtiyoriy funksiya. Qaralayotgan holda $B = \sqrt{A} \cdot \sqrt{B}$ ekanini hisobga olsak $a_{11} = A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2 = \left(\sqrt{A}\frac{\partial \xi}{\partial x} + \sqrt{C}\frac{\partial \xi}{\partial y}\right)^2 = 0$

va bunga asosan $a_{12} = A\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial y} =$

$= \left(\sqrt{A}\ \frac{\partial \xi}{\partial x} + \sqrt{C}\frac{\partial \xi}{\partial y}\right)\left(\sqrt{A}\frac{\partial \eta}{\partial x} + \sqrt{C}\frac{\partial \eta}{\partial y}\right) = 0$ bo‘ladi. Shunday qilib, (4) tenglamada $\frac{\partial^2 u}{\partial \xi^2}$ va $\frac{\partial^2 u}{\partial \xi \partial \eta}$ ning oldidagi koeffitsientlar nolga teng bo‘lib, ikkinchi tartibli hosilalardan $\frac{\partial^2 u}{\partial \eta^2}$ qoladi, butun tenglamani uning oldidagi koeffitsientga $a_{13}$ ga qisqartirish natijasida tenglama

$$\frac{\partial^2 u}{\partial \eta^2} = F_2(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta})\left(F_2 = -\frac{F}{\overline{a}_{22}}\right)$$

ko‘rinishga ega bo‘ladi. Bu tenglama parabolik turdagi tenglamaning **kanonik** shaklidir.

**1-misol.** Noma’lum koeffitsiyentni aniqlash. Quyidagi parabolik tenglama berilgan, $u_t = a \cdot u_{xx}$, $0 < x < 1$, $t > 0$

Boshlang‘ich shart, $u(x,0) = \sin(\pi x)$. Chegaraviy shartlar, $u(0,t) = 0$, $u(1,t) = 0$

Qo‘shimcha shart. $u(1/2,1) = 0.2$. Noma’lum a koeffitsiyentni toping.

Yechimi. Berilgan tenglamaning yechimi. $u(x,t) = e\hat{}(-a\pi^2 t) \sin(\pi x)$. Endi qo‘shimcha shartga qo‘yamiz, $u(1/2,1) = e^{-a\pi^2} \sin(\pi/2)$

Ma’lumki, $\sin(\pi/2) = 1$

Demak, $e^{-a\pi^2} = 0.2$. Har ikki tomondan logarifm olamiz, $-a\pi^2 = \ln(0.2)$

Bundan, $a = -\ln(0.2)/\pi^2$. Yoki, $a = \ln(5)/\pi^2$

Hisoblaymiz, $a = 1.609 / 9.87$, $a \approx 0.163$

**Javob.** $a \approx 0.163$.

Mazkur maqolada parabolik tenglamalarda teskari masalalarning sonli yechimi masalasi o‘rganildi. Parabolik tenglamalarga qo‘yilgan teskari masalalarda noma’lum koeffitsiyentlar, manba funksiyalari yoki chegaraviy shartlarni qo‘shimcha ma’lumotlar asosida aniqlash zarurligi ko‘rsatildi[6]. Bunday masalalar amaliy jihatdan muhim bo‘lishi bilan birga, ko‘pincha nokorrekt qo‘yilgan masalalar turiga kirishi sababli ularni yechishda maxsus sonli usullarni qo‘llash talab etiladi. Maqolada teskari masalalarning matematik qo‘yilishi tahlil qilinib, ularni yechishda ayirmali sxema, approksimatsiya va iteratsion usullarning ahamiyati yoritildi. Keltirilgan misollar orqali noma’lum parametr va manba funksiyalarini qo‘shimcha shartlar asosida aniqlash mumkinligi ko‘rsatildi.

Tadqiqot natijalari shuni ko‘rsatadiki, sonli usullar parabolik tenglamalarga qo‘yilgan teskari masalalarni yechishda samarali vosita hisoblanadi[5]. Ushbu usullar nazariy matematika bilan bir qatorda issiqlik jarayonlari, diffuziya masalalari, geofizika, texnika va boshqa amaliy sohalardagi masalalarni hal etishda keng qo‘llanilishi mumkin.

**Foydalanilgan adabiyotlar ruyxati.**

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