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ANIQ INTEGRALNING GEOMETRIK TATBIQLARI

TOSHKENT DAVLAT
IQTISODIYOT UNIVERSITETI
INTERNATIONAL BUSINESS
AKADEMIK LITSEYI O'QITUVCHISI
ABDUFAYOZOV JAHONGIR FAXRIDDIN O'G'LI

Yuqoridan $y = f(x) \geq 0$ funksiyaning grafigi bilan, yon tomonlardan $x = a$ va $x = b$ vertikal to'g'ri chiziqlar bilan hamda quyidan $y = 0$, ya'ni OX o'qi bilan chegaralangan egri chiziqli trapetsiyaning yuzasi

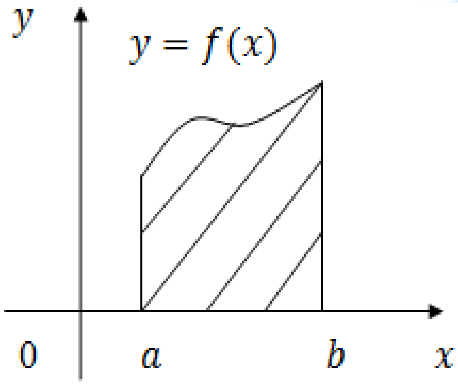
$$S = \int_a^b f(x)dx$$

aniq integral bilan hisoblanishi bizga ma'lum(1-chizma).

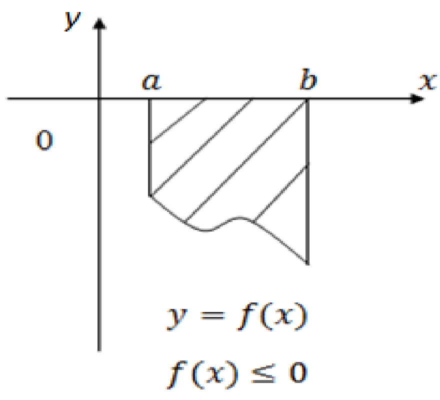
Agar $[a, b]$ kesmada $f(x) \leq 0$ bo'lsa, u holda egri chiziqli trapetsiya OX o'qidan pastda joylashgan bo'lib, uning qiymati manfiy son bo'ladi. Shu sababli, bu holda, egri chiziqli trapetsiya'ning yuzasi

$$S = -\int_a^b f(x)dx = \left|\int_a^b f(x)dx\right|$$

formula bilan topiladi(2-chizma)



1-chizma



2-chizma

$y = f(x)$ va $y = g(x)dx$ $[f(x) \geq g(x)]$ egri chiziqlar hamda $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan geometrik shaklning yuzasi

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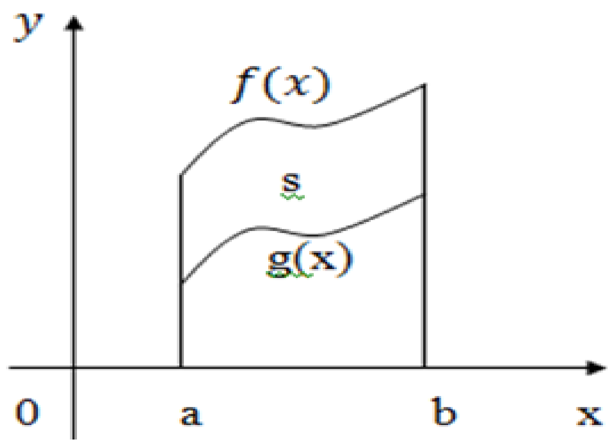
$$S=\int_a^b [f(x)-g(x)]\,dx$$

formula bilan hisoblanadi(3-chizma).



3-chizma

Agar egri chiziq $x=\varphi(t), y=\psi(t)\ (t\epsilon[\alpha;\beta])$ parametrik tenglama bilan berilgan bo‘lsa, u holda egri chiziqli trapetsiyaning yuzasi

$$S=\int_a^b f(x)\,dx=\int_a^b y\,dx=\int_\alpha^\beta \psi(t)d\varphi(t)=\int_\alpha^\beta \psi(t)\,\varphi'(t)\,dt$$

formuladan topiladi.

Tekislikdagi $y=f(x),\ \ x\epsilon[a,b]$ funksiya bilan berilgan egri chiziqning AB yoyi uzunligi

$$l=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

formula bo‘yicha hisoblanadi.

Agar egri chiziq $x=\varphi(t), y=\psi(t)\ (t\epsilon[\alpha;\beta])$ parametrik tenglama bilan berilgan bo‘lsa, u holda, yoy uzunligi

$$l=\int_\alpha^\beta \sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt$$

formula bilan hisoblanadi.

Aytaylik biror jismning OX o‘qiga perpendikulyar bo‘lgan tekislik bilan kesimi yuzi $S(x)$ bo‘lsin. Bu kesim ko‘ndalang kesim deb ataladi va u $[a,b]$ kesmada uzluksizdir. Bu holda, berilgan jismning hajmi

$$V=\int_a^b S(x)dx$$

formula bilan aniqlanadi.

$y=f(x)$ egri chiziq $x=a,\ x=b$ to‘g’ri chiziqlar va OX o‘qi bilan chegaralangan egri chiziqli trapetsiya’ning OX o‘qi atrofida aylanishidan hosil bo‘lgan jismning hajmi

$$V=\pi\int_a^b y^2dx=\pi\int_a^b [f(x)]^2\,dx$$

formuladan, sirti esa

$$S=2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$

formuladan topiladi.

$x=\varphi(y)$ egri chiziq, $y=c, y=d$ to‘g’ri chiziqlar va $0y$ o‘qi bilan chegaralangan egri chiziqli trapetsiya’ning OY o‘qi atrofida aylanishidan hosil bo‘lgan jismning hajmi

$$V=\pi\int_c^d x^2dy=\pi\int_c^d \varphi^2\,(y)dy$$

formuldan topiladi.

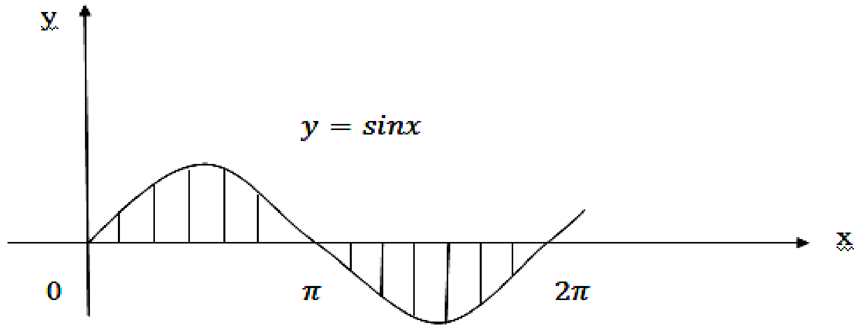
Mavzuga doir yechimlari bilan berilgan topshiriqlardan namunalar

1. $x\epsilon[0;2\pi]$ bo‘lganda, $y=sinx$ sinusoida va OX o‘qi bilan chegaralangan yuza topilsin.

Yechish: $x\epsilon[0;\pi]$ da $sinx\geq 0$ va $x\in[\pi;2\pi]$ da $sinx\leq 0$ bo‘lgani uchun $S=\int_0^\pi sinxdx+|\int_\pi^{2\pi} sinxdx|=-cosx\,|_0^\pi+\left|-cosx\,|_\pi^{2\pi}\right|=$

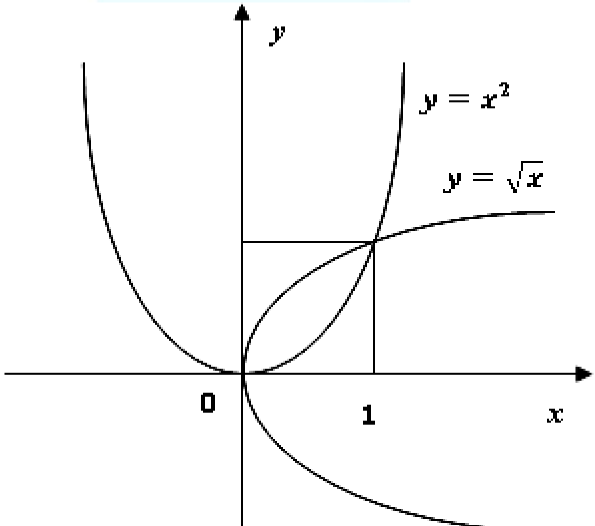
$=-cos\pi+cos0+|-cos2\pi+cos\pi|=1+1+|-1-1|=2+2=4.$

(4-chizma).



2.$y = \sqrt{x}$ va $y = x^2$ egri chiziqlar bilan chegaralangan yuza topilsin.

Yechish: Dastlab $y = \sqrt{x}$ va $y = x^2$ egri chiziqlarni kesishish nuqtalarini topamiz (5-chizma).



5-chizma

$y = \sqrt{x}$ va $y = x^2$ dan $x = x^4$ kelib chiqadi. Undan esa $x_1 = 0$, $x_2 = 1$ larni topamiz. Demak,

$$S = \int_0^1 (\sqrt{x} - x^2)dx = \frac{2}{3}x^{\frac{3}{2}}|_0^1 - \frac{x^3}{3}|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

3. $x = acost, y = bsint$ ellips bilan chegaralangan sohaning yuzi topilsin.

Yechish: Ellipsning yuqori yarim qismini yuzini topamiz va uni ikkiga ko‘paytiramiz. Bu yerda x o‘zgaruvchi $-a$ dan $+a$ gacha o‘zgarganda t o‘zgaruvchi π dan 0 gacha o‘zgaradi. Demak,

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$$S = 2\int_{\pi}^{0} bsint(-asint)dt = -2ab\int_{\pi}^{0} sin^2 t dt = 2ab\int_{0}^{\pi} sin^2 t dt = \quad = 2ab\int_{0}^{\pi}\frac{1-cos2t}{2}dt = ab\int_{0}^{\pi}(1-cos2t)\,dt = ab\left(t-\frac{1}{2}sin2t\right)\Big|_0^{\pi} = \quad = ab\left(\pi - 0 - \frac{1}{2}sin2\pi + \frac{1}{2}sin0\right) = ab(\pi - 0) = \pi ab\,.$$

4. $x^2 + y^2 = r^2$ aylana uzunligi topilsin.

Yechish: Dastlab aylananing birinchi chorakda yotgan bo‘lagining uzunligini topamiz. U holda AB yoy uzunligi $y = \sqrt{r^2 - x^2}$ bo‘ladi va undan esa

$\frac{dy}{dx} = -\frac{x}{\sqrt{r^2-x^2}}$ ni aniqlaymiz. Shunday qilib

$$\frac{1}{4}l = \int_0^r \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = \int_0^r \frac{r}{\sqrt{r^2-x^2}}dx = r\,arcsin\frac{x}{r}\Big|_0^r = r\cdot arcsin1 - -rarcsin0 = r\cdot\frac{\pi}{2} = \frac{\pi r}{2}.$$

Butun aylananing uzunligi esa $l = 4\cdot\frac{\pi r}{2} = 2\pi r$ ga teng bo‘ladi.

5. $x = acos^3 t,\ \ y = asin^3 t$ astroidaning uzunligi topilsin.

Yechish: Egri chiziq har ikkala koordinata o‘qlariga nisbatan simmetrik bo‘lgani uchun dastlab uning to‘rtdan bir qismining uzunligini topamiz. Buning uchun x'_t va y'_t larni topamiz. $x'_t = (acos^3 t)' = -3acos^2 t\cdot sint,$ $y'_t = (asin^3 t)' = 3asin^2 t\cdot cost$ bo‘lib, t parametr 0 dan $\frac{\pi}{2}$ gacha o‘zgaradi. Demak,

$$\frac{1}{4}S = \int_0^{\frac{\pi}{2}}\sqrt{(x'_t)^2 + (y'_t)^2}\,dt == \int_0^{\frac{\pi}{2}}\sqrt{9a^2cos^4 t\cdot sin^2 t + 9a^2 sin^4 t\cdot cos^2 t}\,dt =$$

$$= 3a\int_0^{\frac{\pi}{2}}\sqrt{\cos^2 t\cdot\sin^2 t}\,dt = 3a\int_0^{\frac{\pi}{2}} sint\cdot costdt = \frac{3a}{2}\int_0^{\frac{\pi}{2}} sin2t\,dt =$$

$$= -\frac{3a}{4}cos2t\Big|_0^{\frac{\pi}{2}} = -\frac{3a}{4}(cos\pi - cos0) = -\frac{3a}{4}(-1-1) = \frac{3a}{2}.$$

Demak, $S = 4\cdot\frac{3a}{2} = 6a.$

6. Asos yuzasi S ga teng ko‘pburchak va balandligi H bo‘lgan piramidaning hajmini toping.

Yechish: Geometriya kursidan ma’lumki, piramida asosiga parallel bo‘lgan tekislik bilan kesilsa, kesimda asosiga o‘xshash ko‘pburchak hosil bo‘ladi hamda kesim va asos yuzalarining nisbati ulardan piramida uchigacha bo‘lgan masofalar

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kvadratlarinig nisbati kabi bo‘ladi. Agar piramida asosidan h ga teng masofada asosiga parallel tekislik o‘tkazilganda hosil bo‘lgan kesimning yuzasini $S(h)$ deb olamiz. U holda piramida uchidan kesimgacha masofa $H-h$ bo‘lganligi uchun quyidagiga ega bo‘lamiz:

$$\frac{S(h)}{S}=\frac{(H-h)^2}{H^2};\ S(h)=\frac{S}{H^2}(H-h)^2.$$

Shunday qilib integrallash o‘zgaruvchisi h bo‘lib, u 0 dan H gacha o‘zgaradi. Demak,$V=\int_0^H\frac{S}{H^2}\left(H-h\right)^2dh=\frac{S}{H^2}\int_0^H\left(H^2-2Hh+h^2\right)dh=\frac{S}{H^2}(H^2h--Hh^2+\frac{h^3}{3})\Big|_0^H=\frac{S}{H^2}(H^3-H^3+\frac{H^3}{3})=\frac{S}{H^2}\cdot\frac{H^3}{3}=\frac{SH}{3}$.

7. $y=4x-x^2$ parabola va OX o‘qi bilan chegaralangan shaklning OX o‘qi atrofida aylanishidan hosil bo‘lgan jismning hajmini toping.

Yechish: Dastlab integrallash chegaralarini topamiz. Buning uchun $y=4x-x^2$ va $y=0$ tenglamalarni birgalikda yechamiz. Demak, $4x-x^2=0$ bo‘lib, undan $x_1=0$ va $x_2=4$ kelib chiqadi. Shunday qilib, egri chiziq OX o‘qini ikkita $(0;0)$ va $(4;0)$ nuqtalarda kesib o'tadi va integrallash chegarasi 0 dan 4 gacha bo‘ladi. Izlanayotgan hajm

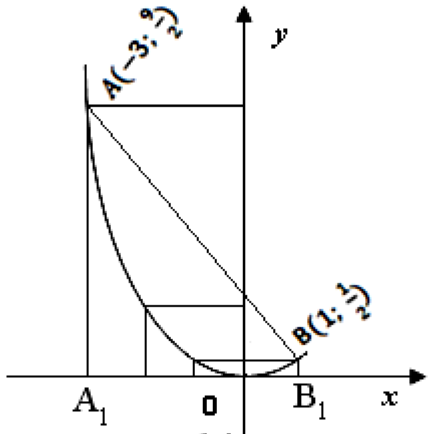
$$V=\pi\int_a^b y^2dx=\pi\int_0^4\ (4x-x^2)^2dx=\pi\int_0^4(16x^2-8x^3+x^4)dx=$$

$$=\pi(\frac{16x^3}{3}-2x^4+\frac{x^5}{5})\Big|_0^4=\pi\left(\frac{16}{3}\cdot 64-2\cdot 4^4+\frac{4^5}{5}\right)=$$

$$=\left(\frac{1024}{3}-512+\frac{1024}{5}\right)\pi=\frac{512}{15}\pi=34{,}2\pi.$$

8. $2y=x^2$ va $2x+2y-3=0$ chiziqlar bilan chegaralangan shaklni OX o‘qi atrofida aylanishidan hosil bo‘lgan jismning hajmi topilsin.

Yechish: $2y=x^2$ dan $y=\frac{1}{2}x^2$ bo‘lib uning grafigi paraboladan iborat. $2x+2y-3=0$ dan $2x+2y=3$ yoki $\frac{x}{1.5}-\frac{y}{1.5}=1$ bo‘lib, u to‘g’ri chiziqdan iborat. Ularni yasaymiz (6-chizma).



6-chizma

Berilgan chiziqlar bilan chegaralangan OAB shaklning ox o‘qi atrofida aylanishidan hosil bo‘lgan jismning hajmi A_1ABB_1 va A_1AOBB_1egri chiziqli trapetsiyalarning ox o‘qi atrofida aylanishidan hosil bo‘lgan jismlar hajmlarining ayirmasidan iborat bo‘ladi. Ularni har birini alohida- alohida topamiz:

$$V_1 = \pi\int_{x_1}^{x_2} y^2dx = \pi\int_{-3}^{1}(1{,}5-x)^2dx = -\pi\int_{-3}^{1}(1.5-x)^2d(1{,}5-x) =$$

$$= -\frac{\pi}{3}(1{,}5-x)^3\Big|_{-3}^{1} = -\frac{\pi}{3}\left(\frac{1}{8}-\frac{729}{8}\right) = \frac{91\pi}{3};$$

$$V_2 = \frac{\pi}{4}\int_{-3}^{1}x^4dx = \frac{\pi}{4}\cdot\frac{x^5}{5}\Big|_{-3}^{1} = \frac{\pi}{20}(1+243) = \frac{\pi}{20}\cdot 244 = \frac{61}{5}\pi$$

Demak, izlanayotgan hajm $V = V_1 - V_2 = \frac{91\pi}{3} - \frac{61\pi}{5} = 18\frac{2}{15}\pi$.

9. $y = x^2$ va $8x = y^2$ parabolalar bilan chegaralangan shaklning oy o‘qi atrofida aylanishidan hosil bo‘lgan jismning hajmi topilsin.

Yechish: $y = x^2$ va $8x = y^2$ parabolalarni yasaymiz. Dastlab ularning kesishish nuqtalarini topamiz (7-chizma).

Buning uchun $y = x^2$ va $8x = y^2$ larni birgalikda yechamiz.

$$\begin{cases} y = x^2 \\ y^2 = 8x \end{cases}$$

Bundan $y_1 = 0$, va $y_2 = 4$ ni topamiz.

Demak,

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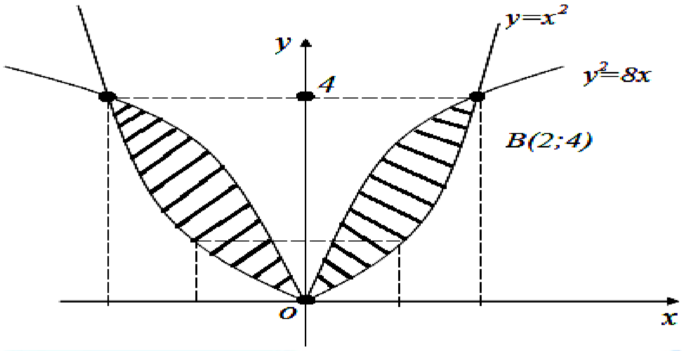
$$V = \pi \int_0^4 \left(y - \frac{y^4}{64}\right) dy = \pi \left(\frac{y^2}{2} - \frac{y^5}{320}\right) \Big|_0^4 = \pi \left(\frac{4^2}{2} - \frac{256 \cdot 4}{320}\right) =$$

$$= \pi \left(8 - \frac{16}{5}\right) = \frac{24\pi}{5}.$$



7-chizma