CHIZIQLI TENGLAMALAR SISTEMASINI KRAMER USULI BILAN YECHISH TADBIQI

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Annotatsiya. Ushbu uslubiy material chiziqli algebraning eng muhim mavzularidan biri — chiziqli tenglamalar sistemasini (ChTS) determinantlar (Kramer qoidasi) yordamida yechishga bag'ishlangan. Matnda ikki va uch noma'lumli chiziqli tenglamalar sistemalarini yechish algoritmlari nazariy hamda amaliy jihatdan to'liq yoritilgan.

Annotation. This methodological material is dedicated to one of the most important topics in linear algebra — solving systems of linear equations (SLE) using determinants (Cramer's rule). The text comprehensively covers both the theoretical and practical aspects of the solution algorithms for systems of linear equations with two and three unknowns.

Kalit so'zlar: Chiziqli tenglamalar sistemasi, Kramer usuli, determinant (aniqlovchi), yordamchi determinant, yagona yechim, cheksiz ko'p yechim, birgalikda bo'lmagan sistema, proporsional koeffitsiyentlar.

Key words: System of linear equations, Cramer's rule, determinant, auxiliary determinant, unique solution, infinitely many solutions, inconsistent system, proportional coefficients.

Chiziqli tenglamalar sistemasini Kramer usuli bilan yechish

Chiziqli tenglamalar sistemasining yechimini topishni oldin ikki noma'lumli ikkita chiziqli tenglamalar sistemasi uchun qaraymiz. Ushbu ikki noma'lumli ikkita chiziqli tenglamalar sistemasi

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

dan, birinchi tenglamani a_{22} ga, ikkinchi tenglamani $-a_{12}$ ga hadma-had ko'paytiramiz va hosil bo'lgan tenglamalarni qo'shamiz, natijada

$$(a_{11}a_{22} - a_{21}a_{12})x = b_1a_{22} - b_2a_{12} \qquad (1)$$

tenglama hosil bo'ladi. Xuddi shunga o'xshash, 1-tenglamani $-a_{21}$ ga, 2-tenglamani a_{11} ga hadma-had ko'paytirib, hosil bo'lgan tenglamalarni qo'shib ushbuni hosil qilamiz:

$$(a_{11}a_{22}-a_{21}a_{12})y=b_2a_{11}-b_1a_{21} \qquad (2)$$

$$(a_{11}a_{22}-a_{21}a_{12})=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad b_1a_{22}-b_2a_{12}=\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix},$$

$$b_2a_{11}-b_1a_{21}=\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

bo'lgani uchun, quyidagi belgilashlarni kiritib

$$\Delta=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \Delta_1=\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \quad \Delta_2=\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

(1) va (2) tengliklarni

$$\Delta x=\Delta_1, \qquad \Delta y=\Delta_2$$

ko'rinishda yozish mumkin. Bundan $\Delta \neq 0$ bo'lsa,

$$x=\frac{\Delta_1}{\Delta}, \quad y=\frac{\Delta_2}{\Delta}$$

bo'ladi, yoki determinantlar orqali yozsak

$$x=\frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad y=\frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Bu formulalarga Kramer formulalari deyiladi, bunda Δ_1 yordamchi determinant Δ determinantning birinchi ustunini ozod hadlar bilan, Δ_2 da esa ikkinchi ustun ozod hadlar bilan almashtiriladi. Δ determinantga tenglamalar sistemasining determinanti deyiladi.

Shunday qilib, berilgan chiziqli tenglamalar sistemasining determinanti 0 dan farqli bo'lsa, sistema yagona yechimga ega bo'ladi.

Endi sistemaning determinanti 0 ga teng, ya'ni

$$\Delta = a_{11}a_{22} - a_{21}a_{12} = 0 \quad \text{ёки} \quad a_{11}a_{22} = a_{21}a_{12}$$

bo'lsin. Bu holda 1-tenglamaning noma'lumlari oldidagi koeffisiyentlari 2-tenglamaning noma'lumlari oldidagi koeffisiyentlariga proporsionaldir. Haqiqatan, koeffisiyentlardan biri, masalan a_{11} noldan farqli bo'lsin deb $\frac{a_{22}}{a_{11}} = \lambda$ bilan belgilasak, bundan $a_{21} = \lambda a_{11}$ bo'ladi. U holda $a_{11}a_{22} = a_{21}a_{12}$ tenglikdan $a_{11}a_{22} = \lambda a_{11}a_{12}$ bo'lib, $a_{22} = \lambda a_{12}$ kelib chiqadi. Bularni hisobga olib, berilgan sistemani

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ \lambda(a_{11}x + a_{12}y) = b_2 \end{cases} \qquad (3)$$

ko'rinishda yozish mumkin. bunda ikkita xususiy hol bo'lishi mumkin:

1) ikkala Δ_1 va Δ_2 determinantlar 0 ga teng, ya'ni $\Delta_1 = b_1a_{22} - b_2a_{12} = 0, \quad \Delta_2 = b_2a_{11} - b_1a_{21} = 0$ bundan $b_2 = \lambda b_1$, chunki $a_{22} = \lambda a_{12}$.

Bu holda $a_{21}, \ a_{22}, \ b_2$ sonlar $a_{11}, \ a_{12}, \ b_1$ sonlarga proporsional bo'lib, berilgan tenglamalar sistemasi ushbu ko'rinishda bo'ladi:

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ \lambda(a_{11}x + a_{12}y) = \lambda b_1 \end{cases}$$

Shunday qilib, sistemaning ikkinchi tenglamasi, birinchi tenglamasidan uning ikkala qismini λ ga ko'paytirish bilan hosil qilinadi, ya'ni u 1-tenglamaning natijasidir. Bu holda berilgan sistema cheksiz ko'p yechimlar to'plamiga ega bo'ladi. Masalan, y ga ixtiyoriy qiymatlar berib, x ning tegishli qiymatini $x = \frac{b_1 - a_{12}y}{a_{11}}$ tenglikdan topamiz.

2) $\Delta_1$va Δ_2 determinantlardan hyech bo'lmaganda bittasi 0 dan farqli, masalan, $\Delta_2 = b_2a_{11} - b_1a_{21} \neq 0$ bo'lsin. U holda $b_2a_{11} \neq b_1a_{21}$ bo'ladi, demak $b_2 \neq \lambda b_1$.

bu holda (3) sistemadan ma'lum bo'ladiki, $\lambda(a_{11}x + a_{12}y) = b_2$ tenglama birinchi $a_{11}x + a_{12}y = b_1$ tenglamaga qarama-qarshidir. Demak, berilgan sistema yechimga ega emas, ya'ni birgalikda emas.

Endi uch noma'lumli uchta tenglamalar sistemasini qaraymiz:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \qquad (4)$$

tenglamalar sistemasi berilgan bo'lsin. Bu sistema noma'lumlari koeffisiyentlaridan ushbu determinantni tuzamiz:

$$\Delta = \begin{vmatrix} a_{11} a_{12} a_{13} \\ a_{21} a_{22} a_{23} \\ a_{31} a_{32} a_{33} \end{vmatrix}$$

bunga (4) ***<u>sistemaning determinanti</u>*** yoki aniqlovchisi deyiladi. $\Delta \neq 0$ bo'lsa, (4) sistema yagona

$$x_1 = \frac{\Delta x_1}{\Delta}, x_2 = \frac{\Delta x_2}{\Delta}, x_3 = \frac{\Delta x_3}{\Delta} \qquad (5)$$

yechimga ega bo'ladi, bunda

$$\Delta x_1 = \begin{vmatrix} b_1 a_{12} a_{13} \\ b_2 a_{22} a_{23} \\ b_3 a_{32} a_{33} \end{vmatrix}, \Delta x_2 = \begin{vmatrix} a_{11} b_1 a_{13} \\ a_{21} b_2 a_{23} \\ a_{31} b_3 a_{33} \end{vmatrix}, \Delta x_3 = \begin{vmatrix} a_{11} a_{12} b_1 \\ a_{21} a_{22} b_2 \\ a_{31} a_{32} b_3 \end{vmatrix}$$

(5) formulaga ham ikki noma'lumli ikkita tenglamalar sistemasidagidek ***Kramer formulalari*** *deyiladi*. Kramer formulalari n noma'lumli n ta tenglamalar sistemasi uchun ham umumlashtiriladi.

Endi misollar qaraymiz:

1-misol. Ushbu

$$\begin{cases} 2x_1 - 3x_2 = 1, \\ 3x_1 + x_2 = 18 \end{cases}$$

tenglamalar sistemasining yechimini toping.

Yechish. Bu sistemaning determinanti

$$\Delta = \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = 2 \cdot 1 - (-3) \cdot 3 = 2 + 9 = 11 \neq 0.$$

Demak, berilgan tenglamalar sistemasi yagona yechimga ega.

$$\Delta x_1 = \begin{vmatrix} 1 & -3 \\ 18 & 1 \end{vmatrix} = 1 + 18 \cdot 3 = 55, \quad \Delta x_2 = \begin{vmatrix} 2 & 1 \\ 3 & 18 \end{vmatrix} = 36 - 3 = 33.$$

Shunday qilib, $x_1 = \dfrac{\Delta x_1}{\Delta} = \dfrac{55}{11} = 5, x_2 = \dfrac{\Delta x_2}{\Delta} = \dfrac{33}{11} = 3.$

2-misol. Ushbu

$$\begin{cases} 2x_1 + 3x_2 - 3x_3 = 3, \\ 4x_1 + 6x_2 - 2x_3 = 6, \\ 3x_1 - x_2 + 2x_3 = -1 \end{cases}$$

tenglamalar sistemasining yechimini toping.

Yechish. Oldin sistemaning determinantini hisoblaymiz:

$$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 6 & -2 \\ 3 & -1 & 2 \end{vmatrix} = 24 - 18 + 4 + 18 - 24 - 4 = 0.$$

Sistema determinanti 0 ga teng, bunda ikki hol bo'lishi mumkin. Tenglamalar sistemasi yechimga ega bo'lmasligi yoki cheksiz ko'p yechimga ega bo'lishi mumkin. Buni aniqlash uchun Δ_1, Δ_2, Δ_3 yordamchi determinantlarni hisoblaymiz:

$$\Delta_1 = \begin{vmatrix} 3 & 3 & -1 \\ 6 & 6 & -2 \\ -1 & -1 & 2 \end{vmatrix} = 0, \quad \Delta_2 = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 6 & -2 \\ 3 & -1 & 2 \end{vmatrix} = 0, \quad \Delta_3 = \begin{vmatrix} 2 & 3 & 3 \\ 4 & 6 & 6 \\ 3 & -1 & -1 \end{vmatrix} = 0$$

Ikkinchi va birinchi tenglamalarni solishtirib, ikkinchi tenglama birinchi tenglamadan ikkiga ko'paytirish bilan hosil bo'lganligini payqaymiz. Demak, berilgan sistema

$$\begin{cases} 2x_1 + 3x_2 - 3x_3 = 3, \\ 3x_1 - x_2 + 2x_3 = -1 \end{cases} \qquad (6)$$

tenglamalar sistemasiga teng kuchli bo'ladi. Bu sistemaning birorta noma'lumiga ixtiyoriy qiymatlar berish bilan cheksiz ko'p yechimlar to'plamiga ega bo'lamiz, masalan,

$x_3 = 1$ bo'lsin, uni oxirgi sistemaga qo'ysak,

$$\begin{cases} 2x_1 + 3x_2 = 4 \\ 3x_1 - x_2 = -3 \end{cases}$$

sistema hosil bo'lib, $x_1 = -\frac{5}{11}$, $x_2 = \frac{18}{11}$ bo'ladi. Bu holda $\left(\frac{-5}{11}, \frac{18}{11}, 1\right)$ yechim hosil bo'ladi. $x_3 = -2$ bo'lsin, buni (6) sistemaga qo'yib, quyidagi sistemani hosil qilamiz:

$$\begin{cases} 2x_1 + 3x_2 = 1, \\ 3x_1 - x_2 = 3 \end{cases}$$

bundan, $x_1 = \frac{10}{11}$, $x_2 = -\frac{3}{11}$ bo'lib, $\left(\frac{10}{11}, -\frac{3}{11}, -2\right)$ yechimni olamiz.

Shunday qilib, noma'lumlarning biriga ixtiyoriy qiymatlar berib, cheksiz ko'p yechimlarni olamiz.

3-misol. Ushbu

$$\begin{cases} x_1 - x_2 + 2x_3 = 2, \\ 2x_1 - 2x_2 + 4x_3 = 4, \\ 3x_1 - 3x_2 + 6x_3 = 3 \end{cases}$$

tenglamalar sistemasini yeching.

Yechish. Berilgan sistema determinantini hisoblaymiz:

$$\Delta = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{vmatrix} = 0$$

bo'lib, yordamchi determinantlar ham $\Delta_1 = \Delta_2 = \Delta_3 = 0$ bo'ladi. Bu tenglamalar sistemasi yechimga ega emas, chunki 1-tenglama bilan 3-tenglama o'zaro ziddir, ya'ni 1-tenglamani -3 ga ko'paytirib 3- tenglamaga hadma-had qo'shsak, 0=-3 tenglik hosil bo'lib, bu tenglamalar sistemasining birgalikda bo'lmasligini bildiradi.

FOYDALANILGAN ADABIYOTLAR

1. Sh.A.Ayupov, B.A.Omirov, A.X.Xudoyberdiyev Chiziqli algebra . Darslik 2023 y.
2. А. С. Мищенко, Эрмитова К-теория. Теория характеристических классов, методы функционального анализа, УМН, 1976, том 31
3. С. П. Новиков , Алгебраическое построение и свойства эрмитовых аналогов Z-теории над кольцами с инволюцией с точки зрения гамильтонового формализма. Некоторые применения к дифференциальной топологии и теории характеристических классов. I, Изв. АН, сер. матем.