MAXSUS FUNKSIYALARNING KAST TARTIBLI HOSILALARDA TATBIQLARI VA ULARNING ASIMPTOTIK XOSSALARI

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Annotatsiya: Ushbu maqolada matematik fizika va kasr tartibli hosilali hisoblashda muhim ahamiyatga ega bo‘lgan maxsus funksiyalar tizimi tadqiq etilgan. Kasr tartibli differensial tenglamalar nazariyasida relaksatsiya, anomal diffuziya va to‘lqin jarayonlarini modellashtirishda fundamental asos hisoblangan Mittag-Leffler funksiyasi, uning ikki parametrli va ko‘p hadli umumlashmalari, shuning siding Prabhakar hamda Rayt funksiyalari tahlil qilingan. Shuningdek, Mellin-Barnes tipidagi integral orqali aniqlanuvchi Foksning H-funksiyasi, uning asimptotik yoyilmalari hamda $z \to 0$ va $|z| \to \infty$ dagi baholari keltirilgan. Mazkur funksiyalar orasidagi analitik bog‘lanishlar kasr hosilali tenglamalarning aniq yechimlarini topish nuqtayi nazaridan asoslab berilgan.

Kalit so‘zlar: Kasr tartibli hosila, maxsus funksiyalar, Mittag-Leffler funksiyasi, Prabhakar funksiyasi, Rayt funksiyasi, Foksning H-funksiyasi, Mellin-Barnes integrali, asimptotik yoyilma, anomal diffuziya

KIRISH

Zamonaviy matematik modellashtirishda chiziqli bo‘lmagan, irsiy xususiyatga va "xotira effekti"ga ega bo‘lgan jarayonlarni tavsiflashda kasr tartibli differensial tenglamalar o‘rni beqiyosdir. Klassik butun tartibli hosilalardan farqli o‘laroq, kasr tartibli anomal diffuziya, dissipativ mexanika va makroiqtisodiy dinamika kabi jarayonlarni ifodalovchi tenglamalar yechimlari elementar funksiyalar (eksponenta, sinus, kosinus) orqali emas, balki maxsus funksiyalar sinfi orqali aniq ifodalanadi.
Ushbu ishda kasr tartubli hosilali hisoblashda va matematik fizikada muhim rol o‘ynovchi Mittag-Leffler, Prabhakar, Rayt va Foksning H-funksiyalarining ta’riflari, asimptotik xossalari hamda ular orasidagi fundamental munosabatlar tahlil qilinadi.

Mittag–Leffler funksiyasi

Mittag–Leffler funksiyasi kompleks o‘zgaruvchining butun funksiyasi bo‘lib, quyidagi darajali qator orqali aniqlanadi:

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \qquad z \in \mathbb{C}, \quad \Re(\alpha) > 0. \tag{1}$$

Ikki parametrli Mittag–Leffler funksiyasi quyidagicha aniqlanadi:

$$E_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)},\qquad z\in\mathbb{C},\quad \Re(\alpha)>0. \tag{2}$$

$\beta=1$ bo‘lganda quyidagi tenglik o‘rinli:

$$E_{\alpha,1}(z)=E_{\alpha}(z).$$

1 Tasdiq. Faraz qilaylik $0<\alpha<2$, $\beta\in\mathbb{R}$ va $\frac{\pi\alpha}{2}<\gamma<\min\{\pi,\pi\alpha\}$. U holda shunday musbat $c=c(\alpha,\beta,\gamma)$ o’zgarmas mavjudki, quyidagi baho o‘rinli:

$$\left|E_{\alpha,\beta}(z)\right|\le\frac{c}{1+|z|},\qquad \gamma\le|\arg z|\le\pi.$$

Umumlashgan Mittag–Leffler funksiyasi (Prabhakar funksiyasi)

U quyidagi qator orqali aniqlanadi:

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\quad z\in\mathbb{C},\quad \Re(\alpha)>0, \tag{3}$$

bu yerda Poxgammer simvoli quyidagicha aniqlanadi:

$$(\gamma)_k=\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)},\quad (\gamma)_0=1. \tag{4}$$

Umumlashgan Mittag–Leffler funksiyasi uchun quyidagi differensial formula o‘rinli:

$$\frac{d}{dz}\left(z^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega z^{\alpha})\right)=z^{\beta-2}E_{\alpha,\beta-1}^{\gamma}(\omega z^{\alpha}). \tag{5}$$

1. Xossa Faraz qilaylik $n\in\mathbb{N}$, $\alpha,\beta,\omega,z\in\mathbb{C}$, $\Re(\alpha)>0$. U holda quyidagi munosabat o‘rinli:

$$E_{\alpha,\beta}^{n+1}(-\omega z^{\alpha})-E_{\alpha,\beta}^{n}(-\omega z^{\alpha})=-\omega z^{\alpha}E_{\alpha,\beta+\alpha}^{n+1}(-\omega z^{\alpha}) \tag{6}$$

Ko‘p hadli Mittag–Leffler funksiyasi

Ko‘p hadli Mittag–Leffler funksiyasi quyidagi qator orqali aniqlanadi

$$E_{(\beta_1,\dots,\beta_m),\beta_0}(z_1,\dots,z_m)=\sum_{k=0}^{\infty}\sum_{k_1+\dots+k_m=k}\frac{k!}{k_1!\cdots k_m!}\frac{\prod_{j=1}^{m}z_j^{k_j}}{\Gamma\left(\beta_0+\sum_{j=1}^{m}\beta_j k_j\right)}, \tag{7}$$

bu yerda

$$0<\beta_0<2,\qquad 0<\beta_j<1\quad (j=1,\dots,m),\qquad z_j\in\mathbb{C},$$

$k_j\in\mathbb{N}_0$, $k=\sum_{j=1}^{m}k_j$, va

$$\frac{k!}{k_1!\cdots k_m!}$$

ko‘paytuvchi multinomial koeffitsiyentni ifodalaydi.

2. Tasdiq. Faraz qilaylik $0<\beta<2$ va $1>\alpha_1>\dots>\alpha_m>0$. Quyidagi shartlar bajarilsin: $\frac{\alpha_1\pi}{2}<\mu<\alpha_1\pi$, $\mu\le|\arg z_1|\le\pi$, hamda shunday $K>0$ son mavjud bo‘lsinki, $-K\le z_j<0$, $\quad j=2,\dots,m$. U holda shunday musbat $C=C(\mu,K,\alpha_1,\dots,\alpha_m,\beta)$ doimiy mavjud bo‘lib, quyidagi baho o‘rinli:

$$\left|E_{(\alpha_1,\alpha_1-\alpha_2,\dots,\alpha_1-\alpha_m),\beta}(z_1,\dots,z_m)\right| \le \frac{C}{1+|z_1|}. \tag{8}$$

Rayt funksiyasi

Umumlashgan Rayt funksiyasi ${}_p\Psi_q(z)$ quyidagicha aniqlanadi:

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(a_l,\alpha_l)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix} \mid z\right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p}\Gamma\left(a_l+\alpha_l k\right)}{\prod_{j=1}^{q}\Gamma\left(b_j+\beta_j k\right)}\frac{z^k}{k!}. \tag{9}$$

Bu yerda a_l, b_j kompleks sonlar, α_l, β_j esa haqiqiy parametrlar hisoblanadi.

Mazkur funksiya klassik Rayt funksiyasining umumlashmasi bo‘lib, ko‘pincha Foks–Rayt funksiyasi deb ataladi. U Foks, Rayt va boshqa tadqiqotchilar tomonidan o‘rganilgan bo‘lib, katta z qiymatlarida uning asimptotik yoyilmalari olingan.

Xususan, ${}_p\Psi_q(z)$ funksiyaning $|z|\to\infty$ dagi asimptotikasi quyidagi shart bajarilganda mavjud:

$$\sum_{j=1}^{q}\beta_j - \sum_{l=1}^{p}\alpha_l > -1.$$

Rayt funksiyasi Mittag–Leffler funksiyasi orqali quyidagicha ifodalanadi:

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)}\,{}_1\Psi_1[z\left|\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix}\right]. \tag{10}$$

Foksning H-funksiyasi.

Foksning H-funksiyasi kasr tartibli hosilali hisoblashda uchraydigan maxsus funksiyalardan biri bo‘lib, u Mittag–Leffler funksiyasini xususiy hol sifatida o‘z chiga oladi.

U Foks tomonidan Meyjerning G-funksiyasining umumlashmasi sifatida kiritilgan va turli tatbiqlarda muhim rol o‘ynaydi.

Mazkur ishda H-funksiyaning aniqlanishi va asosiy xossalari, belgilashlar biroz o‘zgartirilgan holda qo‘llaniladi.

Foksning $H(z)$ funksiyasi Mellin–Barnes tipidagi integral orqali aniqlanadi:

$$H(z) = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \Theta(s) z^{-s}\,ds, \tag{11}$$

bu yerda

$$\Theta(s) = \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_j s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_j s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-B_j s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+A_j s\right)}, \tag{12}$$

L — Mellin–Barnes tipidagi kontur bo‘lib, u quyidagi gamma-funksiyalar qutblarini ajratadi:

$$0\le n\le p,\qquad 0\le m\le q,\qquad A_j,B_j\in\mathbb{R}_+,\qquad a_j,b_j\in\mathbb{C}.$$

Kontur L quyidagi qutblarni ajratadi:

$$\zeta_{j\nu} = -\frac{b_j + \nu}{B_j}, \tag{13}$$

va

$$\omega_{\lambda k} = \frac{1 - a_\lambda + k}{A_\lambda}, \tag{14}$$

hamda quyidagi shart bajariladi:

$$A_\lambda(b_j + \nu) \neq B_j(a_\lambda - k - 1). \tag{15}$$

Quyidagi belgilashlar kiritiladi:

$$\alpha = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j, \tag{16}$$

$$\beta = \left(\prod_{j=1}^{p} A_j^{-A_j}\right)\left(\prod_{j=1}^{q} B_j^{B_j}\right), \tag{17}$$

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j, \tag{18}$$

$$\delta = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}. \tag{19}$$

Faraz qilaylik (15) shart bajarilgan bo‘lsin. Agar (13) dagi $s = \zeta_{j\nu}$ nuqtalardagi barcha qutblar sodda bo‘lsa, ya’ni quyidagi shart bajarilsa

$$B_j(b_r + k) \neq B_r(b_j + l), \quad r \neq j,\ r, j = 1, \dots, m,\ k, l \in \mathbb{N}_0, \tag{20}$$

va quyidagi shartlardan biri bajarilsa:

$$\mu > 0,\ z \neq 0, \qquad \text{yoki} \qquad \mu = 0,\ 0 < |z| < \beta,$$

u holda (11)dagi H-funksiya quyidagi yoyilmaga ega:

$$H_{p,q}^{m,n}(z) = \sum_{j=1}^{m} \sum_{l=0}^{\infty} h_{jl}^{*}\, z^{\frac{b_j + l}{B_j}}, \tag{21}$$

bu yerda

$$h_{jl}^{*} = \frac{(-1)^l}{l!\ B_j} \frac{\prod_{\substack{k=1 \\ k\neq j}}^{m} \Gamma\left(b_k - (b_j + l)\frac{B_k}{B_j}\right) \prod_{k=1}^{n} \Gamma\left(1 - a_k + (b_j + l)\frac{A_k}{B_j}\right)}{\prod_{k=n+1}^{p} \Gamma\left(a_k - (b_j + l)\frac{A_k}{B_j}\right) \prod_{k=m+1}^{q} \Gamma\left(1 - b_k + (b_j + l)\frac{B_k}{B_j}\right)}. \tag{22}$$

Agar (14) dagi $s = \omega_{\lambda k}$ nuqtalardagi barcha qutblar sodda bo‘lsa, ya’ni quyidagi shart bajarilsa

$$A_j(1 - a_r + k) \neq A_r(1 - a_j + l), \quad r \neq j,\ r, j = 1, \dots, n,\ k, l \in \mathbb{N}_0,$$

va quyidagi shartlardan biri bajarilsa:

$$\mu < 0,\ z \neq 0, \qquad \text{yoki} \qquad \mu = 0,\ |z| > \beta,$$

u holda (11) dagi H-funksiya quyidagi yoyilmaga ega:

$$H_{p,q}^{m,n}(z) = \sum_{r=1}^{n}\sum_{k=0}^{\infty} h_{rk}\ z^{-\frac{a_r-1-k}{A_r}}, \qquad (23)$$

bu yerda

$$h_{rk} = \frac{(-1)^k}{k!\ A_r}\frac{\prod_{j=1}^{m}\Gamma\left(b_j + (1-a_r+k)\frac{B_j}{A_r}\right)\prod_{\substack{j=1\\ j\neq r}}^{n}\Gamma\left(1-a_j-(1-a_r+k)\frac{A_j}{A_r}\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-(1-a_r+k)\frac{A_j}{A_r}\right)\prod_{j=m+1}^{q}\Gamma\left(1-b_j-(1-a_r+k)\frac{B_j}{A_r}\right)}. \qquad (24)$$

(21) munosabat H-funksiyaning (11) dagi $z \to 0$ dagi darajali asimptotik yoyilmasini beradi.

Agar (15) va (20) shartlar bajarilgan bo‘lsa, hamda $\mu \geq 0$ yoki $\mu < 0$ va $\alpha > 0$ bo‘lsa, u holda asimptotikaning asosiy hadlari quyidagi ko‘rinishga ega:

$$H_{p,q}^{m,n}(z) = \sum_{j=1}^{m}\left[h_j^* z^{\frac{b_j}{B_j}} + o\left(z^{\frac{b_j}{B_j}}\right)\right], \qquad (z \to 0), \qquad (25)$$

qo‘shimcha shart ostida

$$|\arg z| < \frac{\alpha\pi}{2}, \quad \text{agar } \mu < 0 \text{ va } \alpha > 0,$$

bu yerda

$$h_j^* = h_{j0}^* = \frac{1}{B_j}\frac{\prod_{\substack{r=1\\ r\neq j}}^{m}\Gamma\left(b_r - b_j\frac{B_r}{B_j}\right)\prod_{r=1}^{n}\Gamma\left(1-a_r+b_j\frac{A_r}{B_j}\right)}{\prod_{r=n+1}^{p}\Gamma\left(a_r-b_j\frac{A_r}{B_j}\right)\prod_{r=m+1}^{q}\Gamma\left(1-b_r+b_j\frac{B_r}{B_j}\right)}. \qquad (26)$$

(23) munosabat H-funksiyaning (11)dagi $|z| \to \infty$ dagi asimptotik yoyilmasini beradi.

Agar (15) va (20) shartlar bajarilgan bo‘lsa, hamda $\mu \leq 0$ yoki $\mu > 0$ va $\alpha > 0$ bo‘lsa, u holda quyidagicha:

$$H_{p,q}^{m,n}(z) = \sum_{r=1}^{n}\left[h_r z^{\frac{a_r-1}{A_r}} + o\left(z^{\frac{a_r-1}{A_r}}\right)\right], \qquad (|z| \to \infty), \qquad (27)$$

qo‘shimcha shart ostida

$$|\arg z| < \frac{\alpha\pi}{2}, \quad \text{agar } \mu > 0 \text{ va } \alpha > 0,$$

bu yerda

$$h_r = h_{r0} = \frac{1}{A_r}\frac{\prod_{j=1}^{m}\Gamma\left(b_j + (1-a_r)\frac{B_j}{A_r}\right)\prod_{\substack{j=1\\ j\neq r}}^{n}\Gamma\left(1-a_j-(1-a_r)\frac{A_j}{A_r}\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-(1-a_r)\frac{A_j}{A_r}\right)\prod_{j=m+1}^{q}\Gamma\left(1-b_j-(1-a_r)\frac{B_j}{A_r}\right)}.$$

(28)

Ushbu asimptotik bahoning bosh hadi quyidagi ko‘rinishda ifodalanadi:

$$H_{p,q}^{m,n}(z) = O\left(z^{\rho^*}\right), \qquad \left(\rho^* := \min_{1\le r\le n}\left[\frac{\Re(a_r) - 1}{\alpha_r}\right]\right), \qquad (|z| \to \infty).$$

Kasr Tartibli Hosilali Modellarning Amaliy Tatbiqlari

Mazkur maxsus funksiyalar shunchaki mavhum qatorlar bo‘lmay, fizik va texnik jarayonlarning matematik modellarida aniq munosabatlarni ifodalaydi.

1. **Anomal Diffuziya va Gidrodinamika:** G‘ovakli va fraktal muhitlarda moddalarning sizib o‘tish tezligi klassik Fik qonuniga bo‘ysinmaydi. Vaqt bo‘yicha va koordinata bo‘yicha kasr tartibli diffuziya tenglamalarining fundamental yechimlari (Grinfild funksiyalari) to‘g‘ridan-to‘g‘ri Foksning H-funksiyasi va Rayt funksiyasi orqali yoziladi.
2. **Yopishqoq-elastiklik (Viscoelasticity):** Polimer va kompozit materiallarning kuchlanish relaksatsiyasi va deformatsiya jarayonlaridagi dinamik xotirani modellashtirishda Prabhakar hamda ikki parametrli Mittag-Leffler funksiyalari qo‘llaniladi. Ular eksperimental ma'lumotlarni minimal xatolik bilan tavsiflaydi.
3. **Avtomatlashtirilgan Boshqaruv:** Zamonaviy sanoatda $PI^{\lambda}D^{\mu}$ (kasr tartibli integral va hosilali) regulyatorlar o‘tish jarayonlarining tebranishlarini kamaytirishda juda samarali. Ushbu tizimlarning barqarorlik va dinamik xarakteristikalari Mittag-Leffler funksiyasining asimptotik baholari (3) va (8) yordamida kafolatlanadi.

Xulosa

Maxsus funksiyalar, xususan Mittag-Leffler, Prabhakar va Foks H-funksiyalari kasr tartibli analiz tilining alifbosi hisoblanadi. Ularning asimptotik va differensial xossalari murakkab fizik-mexanik tizimlarni barqarorlashtirish va bashorat qilish jarayonlarini tubdan osonlashtiradi. Kelajakda ushbu funksiyalarni hisoblashning sonli algoritmlarini hamda optimal kvadratura formulalarini yaratish amaliy muhandislik masalalarini yechishda muhim qadam bo‘ladi.

Adabiyotlar

1. Kasr tartibli hisoblash va differensial tenglamalar nazariyasiga oid tadqiqotlar to‘plami va maxsus funksiyalar tahlili qo‘lyozmasi, 2026. (Matn ichidagi havolalar manbasi).
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