YUKLANGAN BIR O'LCHAMLI ISSIQLIK TARQALISH TENGLAMASI UCHUN BOSHLANG`ICH-CHEGARAVIY MASALA

НАЧАЛЬНО-КРАЕВАЯ ЗАДАЧА ДЛЯ ОДНОМЕРНОГО УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С ИНВОЛЮЦИЕЙ

AN INITIAL-BOUNDARY VALUE PROBLEM FOR ONE DIMENSIONAL HEAT EQUATION WITH INVOLUTION

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Annotatsiya. Ushbu maqolada chegaralangan bir o'lchamli sohada yuklanganlik bilan berilgan issiqlik tarqalish tenglamasi uchun boshlang'ich-chegaraviy masala ko'rib chiqiladi. To'g'ri masala bu tenglama uchun Kosh tipidagi boshlang'ich shartlar va Dirixle chegaraviy shartlar bilan berilgan boshlang'ich-chegaraviy masaladir. O'zgaruvchilarni ajratish usuli bilan berilgan masala ikkinchi turdagi Volterra integral tenglamasiga keltiriladi. Yechimning mavjudligi va yagonaligini isbotlash uchun qisqaruvchi akslantirishlar prinsipiga tayaniladi.

Аннотация. В данной статье рассматривается начально-краевая задача для уравнения теплопроводности в ограниченной одномерной области, заданного с нагрузкой (или с условием нагрузки). Прямая задача представляет собой начально-краевую задачу для данного уравнения с начальными условиями типа Коши и граничными условиями Дирихле. Задача, поставленная методом разделения переменных, сводится к интегральному уравнению Вольтерра второго рода. Для доказательства существования и единственности решения используется принцип сжимающих отображений.

Abstract In this paper, we consider an initial-boundary value problem for a heat equation with involution in a bounded one-dimensional domain. The direct problem is an initial boundary value problem for this equation with Cauchy-type initial data and Drichlet boundary conditions. The problem given by the method of separation of variables is reduced to the Volterra integral equation of the second kind. The principle of contraction mappings is used to prove the existence and uniqueness of the solution.

Kalit so'zlar: issiqlik tarqalish tenglamasi, boshlang'ich-chegaraviy masala, Volterra integral tenglamasi, Granuolla tengsizligi, yuklanganlik.

Ключевые слова: уравнение теплопроводности, начально-краевая задача, интегральное уравнение Вольтерра, неравенство Грануолла, инволютция.

Key words: heat equation, an initial-boundary problem, Volterra integral equation, Gronwall`s inequality,involution.

Consider the non-homogeneous heat equation

$u_t(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) + q(t)u(x,t) = f(x,t),\ (x,t)\epsilon\ \Omega$ (1)

initial condition

$u(x, 0) = \varphi(x),\ x\ \epsilon\ [-\pi, \pi]$ (2)

boundary conditions

$u(-\pi, t) = u(\pi, t) = 0,\ t\ \epsilon\ [0, T]$ (3)

Where $f(x,t), \varphi(x)$ are given and enough smooth functions, ε is a nonzero real number such that $|\varepsilon| < 1$ and Ω is a rectangular domain given by $\Omega = \{-\pi < x < \pi, 0 < t < T\}$. Suppose that the data of the problem (1)-(3) the functions $\varphi(x)$, $f(x, t)$ satisfy following assumption:

A1) $\varphi(x)\epsilon C^2[-\pi,\pi],\ ,\ \varphi^{(3)}(x)\epsilon L[-\pi,\pi]\varphi(-\pi) = \varphi(\pi) = 0$;

A2) $f(x, t)\ \epsilon C(\Omega),\ f(x, t)\ \epsilon L^2\ (\Omega)$.

Here we seek solution to problem in a form of series expansion using a set of functions that form orthogonal basis in $L^2\ (-\pi, \pi)$. To find the appropriate set of functions for each problem, we shall solve the homogeneous equation corresponding to equation (1) along with the associated boundary conditions using separation of variables.

Separation of variables leads to the following spectral problem for (1)-(3)

$X''(x) - \varepsilon X''(-x) + \lambda X(x) = 0\ X(\pi) = X(-\pi) = 0,$ (4)

The eigenvalue problem is self-adjoint and hence they have real eigenvalues and their eigenfunctions form a complete orthogonal basis in $L^2(-\pi, \pi)$. Their eigenvalues are, respectively, given by

$\lambda_{1k}=(1-\varepsilon)(k+\frac{1}{2})^2,\ \ k\epsilon\mathbb{N}\cup\{0\},\ \lambda_{2k}=(1+\varepsilon)k^2,\ k\epsilon\mathbb{N}$ (5)

and the corresponding eigenfunctions are given by

$$X_{1k}(x) = \sqrt{\frac{1}{\pi}} cos\, cos\, (k+\frac{1}{2})\, x,\ \ k\epsilon\mathbb{N} \cup \{0\},\ \ X_{2k}(x) = \sqrt{\frac{1}{\pi}} sin\, sin\, kx\ , k\epsilon\mathbb{N} \quad (6)$$

Lemma 1. *The systems of functions (7) are complete and orthogonal in* $L^2\ (-\pi, \pi)$.

By applying the Fourier method, the solution $u(x, t)$ of the problem (1)-(3) can be expanded in a uniformly convergent series in term of eigenfunctions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)X_{1k}(x)+\sum_{k=1}^{\infty} u_{2k}(t)X_{2k}(x) \quad (7)$$

The coefficients $u_{1k}(t)$, $u_{2k}(t)$ are to be found by making use of the orthogonality of the eigenfunctions. Namely, we multiply (1) by the eigenfunctions of (6) and integrate over $(-\pi,\ \pi)$. Recall that the scalar product in $L^2(-\pi,\ \pi)$ is defined by $(f,\ g) = \int_{-\pi}^{\pi} f(x)g(x)dx$. Let us note the

expansion coefficients of $f(x,\ t)$ and $\varphi(x)$ in the eigenfunctions of (6) for $k \geq 1$ respectively by

$\{(f(x,t),X_{1k}(x)) = f_{1k}(t), (f(x,t),X_{2k}(x)) = f_{2k}(t),$ $\{(\varphi(x,t),X_{1k}(x)) = \varphi_{1k}\ (\varphi(x,t),X_{2k}(x)) = \varphi_{2k}$

In view of (1) for $(u(x,\ t),\ X_{1k}(x)) = u_{1k}(t)$, $(u(x,\ t),\ X_{2k}(x)) = u_{2k}(t)$, we obtain the Cauchy type problems

$$\{u'_{1k}(t) + (1-\varepsilon)(k+\frac{1}{2})^2 u_{1k}(t) + q(t)u_{1k}(t) = f_{1k}(t), u_{1k}(0) = \varphi_{1k}, \tag{8}$$

$$\{u'_{2k}(t) + (1+\varepsilon)k^2 u_{2k}(t) + q(t)u_{2k}(t) = f_{2k}(t), u_{2k}(0) = \varphi_{2k}, \tag{9}$$

We consider the Cauchy problems (8) and (9). During the investigation of Cauchy problems (8) and (9), we obtain an equivalent Volterra integral equations of the second kind:

$$u_{1k}(t)=\varphi_{1k}e^{-(1-\varepsilon)(k+\frac{1}{2})^2 t} + \int_0^t f_{1k}(\tau)e^{-(1-\varepsilon)(k+\frac{1}{2})^2(t-\tau)}d\tau - \int_0^t q(\tau)u_{1k}(\tau)e^{-(1-\varepsilon)(k+\frac{1}{2})^2(t-\tau)}d\tau, \tag{10}$$

$$u_{1k}(t) = \varphi_{2k}e^{-(1+\varepsilon)k^2 t} + \int_0^t f_{2k}(\tau)e^{-(1+\varepsilon)k^2(t-\tau)}d\tau - \int_0^t q(\tau)u_{2k}(\tau)e^{-(1+\varepsilon)k^2(t-\tau)}d\tau \tag{11}$$

For integral equations (10) and (1), the following holds:

Lemma 2. Let $q(t)\epsilon C[0,T], f_{ik}\epsilon C[0,T], i = 1,2$. There holds the following estimate for all $k\ \epsilon$ N and $t\ \epsilon\ [0,\ T]$:

$$|u_{1k}(t)|\leq \left(|\varphi_{1k}| + \frac{||f_{1k}||_{C[0,T]}}{(1-\varepsilon)(k+\frac{1}{2})^2}\right) e^{\frac{||q||}{(1-\varepsilon)(k+\frac{1}{2})^2}} e^{\frac{||q||}{(1-\varepsilon)(k+\frac{1}{2})^2}e^{-(1-\varepsilon)(k+\frac{1}{2})^2 t}}, \tag{12}$$

$$|u_{2k}(t)|\leq \left(|\varphi_{2k}| + \frac{||f_{2k}||_{C[0,T]}}{(1+\varepsilon)k^2}\right) e^{\frac{||q||}{(1+\varepsilon)k^2}} e^{\frac{||q||}{(1+\varepsilon)k^2}e^{-(1+\varepsilon)k^2 t}}, \tag{13}$$

$$|u'_{1k}(t)|\leq ||f_{1k}||_{C[0,T]}+\left(||q||_{C[0,T]} + (1-\varepsilon)(k+\tfrac{1}{2})^2\right)$$
$$\text{x}\left(|\varphi_{1k}| + \frac{||f_{1k}||_{C[0,T]}}{(1-\varepsilon)(k+\frac{1}{2})^2}\right) e^{\frac{||q||}{(1-\varepsilon)(k+\frac{1}{2})^2}} e^{\frac{||q||}{(1-\varepsilon)(k+\frac{1}{2})^2}e^{-(1-\varepsilon)(k+\frac{1}{2})^2 t}}, \tag{14}$$

$|u'_{2k}(t)| \leq ||f_{2k}||_{C[0,T]} + \left(||q||_{C[0,T]} + (1+\varepsilon)k^2\right)$

$\text{x}\left(|\varphi_{2k}| + \frac{||f_{2k}||_{C[0,T]}}{(1+\varepsilon)k^2}\right) e^{\frac{||q||}{(1+\varepsilon)k^2}} e^{\frac{||q||}{(1+\varepsilon)k^2}} e^{-(1+\varepsilon)k^2 t}$ (15)

Theoem 1. If $q(t)\epsilon C[0,T]$ (A1)-(A2) are saisfied, then there exists a unique soltion to the direct problem (1)-(3) $u(x,t) \in C^{2,1}(\Omega)$.

Lemma 3. If the condition (A1) then there are equalities

$\varphi_{1k} = \frac{1}{(k+\frac{1}{2})^3}\varphi'''_{1k}, \ \varphi_{2k} = \frac{1}{k^3}\varphi'''_{2k}, \quad f_{1k}(t) = \frac{1}{k} f_{1k,x}(t), \ f_{2k}(t) = \frac{1}{k} f_{2k,x}(t)$

(16)

where

$\varphi'''_{1k} = -\sqrt{\frac{1}{\pi}}\int_{-\pi}^{\pi} \quad \varphi'''(x) \sin\sin\left(k+\frac{1}{2}\right)x\,dx$, $\varphi'''_{2k} = -\sqrt{\frac{1}{\pi}}\int_{-\pi}^{\pi} \quad \varphi'''(x)$ $\cos\cos kx\,dx$,

$f_{1k,x}(t) = -\sqrt{\frac{1}{\pi}}\int_{-\pi}^{\pi} \quad \left(k+\frac{1}{2}\right)x\,dx$,

$f_{2k,x}(t) = -\sqrt{\frac{1}{\pi}}\int_{-\pi}^{\pi} \quad f_x(x,t)\cos\cos kx\,dx$

with the following estimate:

$\sum_{k=1}^{\infty} \quad |\varphi'''_{1k}|^2 \leq ||\varphi'''||^2_{L_2(-\pi,\pi)}, \ \sum_{k=1}^{\infty} \quad |f_{1k,x}(t)|^2 \leq ||f_x||^2_{L_2(-\pi,\pi)},$

$\sum_{k=1}^{\infty} \quad |\varphi'''_{2k}|^2 \leq ||\varphi'''||^2_{L_2(-\pi,\pi)}, \ \sum_{k=1}^{\infty} \quad |f_{2k,x}(t)|^2 \leq ||f_x||^2_{L_2(-\pi,\pi)},$ (17)

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